# Calculus I Review III Worksheet

Math 140: Calculus with Analytic Geometry

**Name:** _____ **Date:** _____

**Directions.** Show enough work to support each answer. Give exact values. No calculators or other electronic devices are allowed.

**1. Related rates.**

A baseball diamond is a square with side length 90 ft. A runner moves along the straight baseline from first base toward second base at 20 ft/s. Let $x$ be the runner's distance from first base and $y$ the runner's straight-line distance from home plate, both in feet.

(a) Draw and label the diamond, the runner, and the segment from home plate to the runner. Use the Pythagorean Theorem to relate $x$ and $y$. State $dx/dt$, including its sign.

(b) Differentiate your equation with respect to time before substituting the runner's position. How fast is the runner's distance from home plate changing when the runner is $\frac{135}{2}$ ft from first base? Include units and explain the sign of your answer.

(c) As the runner approaches second base, what value does $dy/dt$ approach? Compare this with the runner's speed of 20 ft/s and explain why the two rates differ.

**2. Linearization and differentials.**

Let $f(x) = \arctan x$, with angles measured in radians.

(a) Find the linearization $L(x)$ of $f$ at $a = 1$. Use it to approximate $\arctan(1.04)$. Keep $\pi$ in your answer.

(b) Take $x = 1$ and $dx = 0.04$. Find the differential $dy = f'(x)\,dx$ and write an exact expression for the actual change $\Delta y = f(x + dx) - f(x)$. Explain how $dy$ and $L(1.04)$ are related and why $dy$ need not equal $\Delta y$.

(c) Compute $f''(x)$ and use concavity near $x = 1$ to decide whether the approximation in part (a) is an overestimate or an underestimate. Does $dy$ overestimate or underestimate $\Delta y$?

**3. Rolle's Theorem and the Mean Value Theorem.**

Let $g(x) = x^2(4 - x)$, with domain $\mathbb{R}$.

(a) State Rolle's Theorem, including all hypotheses. Verify those hypotheses for $g$ on $[0, 4]$, then find every $c \in (0, 4)$ that satisfies its conclusion.

(b) State the Mean Value Theorem, including all hypotheses. Apply it to $g$ on $[0, 2]$: verify the hypotheses, compute the average rate of change, and find every $c \in (0, 2)$ where the instantaneous rate equals that average rate. Interpret the equality in terms of tangent and secant slopes.

(c) Explain how Rolle's Theorem is a special case of the Mean Value Theorem. A student keeps $c = 2$ as an answer in part (b) because it satisfies the derivative equation. Explain the error. Does $g'(2) \neq 0$ prevent the Mean Value Theorem from applying on $[0, 2]$?

**4. First and second derivative tests.**

Let $p(x) = \sin x + \dfrac{x}{2}$ on $[-\pi, \pi]$. Use radians.

(a) Find $p'(x)$ and all critical numbers in $(-\pi, \pi)$. Make a sign chart for $p'$ and give the open intervals of increase and decrease. Use the first derivative test to classify every interior critical number as a local maximum, a local minimum, or neither. Give each interior local extremum as a point.

(b) Find $p''(x)$ and use the second derivative test to check your classifications from part (a). Show the second derivative value used in each classification.

(c) Determine the intervals of concavity and all inflection points. At $x = 0$, is the tangent horizontal? Does $p$ have a local extremum there? Explain why a change in concavity need not be a change from increasing to decreasing or vice versa.

**5. Curve sketching and limits at infinity.**

Let $q(x) = \dfrac{\ln x}{x}$, with its natural domain.

(a) State the domain and find all intercepts. Evaluate the limits at both ends of the domain and identify all vertical and horizontal asymptotes. For the limit at infinity, identify the indeterminate form and show an application of L'Hôpital's Rule.

(b) Compute and factor $q'(x)$ and $q''(x)$. Use sign analysis to find intervals of increase, decrease, and concavity, and locate all local extrema and inflection points. Use the second derivative test to check the classification of each critical number.

(c) Sketch the graph, labeling intercepts, asymptotes, extrema, and inflection points. Does $q$ have an absolute maximum or minimum? Justify your conclusions on the entire domain. Does the graph cross its horizontal asymptote?

**6. Optimization on an unbounded domain.**

A rectangle lies in the first quadrant with one vertex at the origin, two sides on the coordinate axes, and its opposite vertex $(x, y)$ on the curve $y = 1/(1 + x^2)$. Its sides are parallel to the axes. Coordinates are measured in meters. Find the rectangle of maximum area.

(a) Draw a labeled sketch and define your variables. Write a mathematical model with area as the objective and include the curve and positivity constraints. Explain why the entire rectangle lies beneath the curve.

(b) Reduce the area to a function $A(x)$ of one variable and state its feasible domain. Compute $A'(x)$, find all feasible critical numbers, and use the second derivative test to classify them locally.

(c) Use derivative signs on the entire feasible domain to justify an absolute maximum. Also evaluate the limits of $A(x)$ as $x \to 0^+$ and $x \to \infty$ and interpret them geometrically. Give the maximizing width, height, and area with units. Explain why the local second derivative test alone would not establish the absolute maximum.