# Riemann Sums

Math 140: Calculus with Analytic Geometry

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## 1 Introduction

In the previous lectures, we approximated definite integrals using left endpoint rectangles, right endpoint rectangles, and midpoint rectangles. In each case, the approximation was obtained by adding the areas of rectangles of the form

$$f(x_i)\Delta x.$$

The main point of this lecture is that all of these constructions are special cases of a more general idea called a *Riemann sum*.

For simplicity, we will assume throughout this lecture that the interval is divided into subintervals of equal width. There is a more general definition in which the subintervals need not all have the same width, but the constant-width case is enough for the examples we consider here.

## 2 The General Construction

Let $f$ be a function on the interval $[a, b]$. We divide $[a, b]$ into $n$ subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

We define the partition points by

$$x_i = a + i\Delta x, \qquad i = 0, 1, 2, \ldots, n.$$

Thus,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

On each subinterval $[x_{i-1}, x_i]$, we choose a sample point $c_i$ satisfying

$$x_{i-1} \le c_i \le x_i.$$

The corresponding rectangle has width $\Delta x$ and height $f(c_i)$, so its area is

$$f(c_i)\Delta x.$$

This leads to the following definition.

**Definition**

A *Riemann sum* for $f$ on $[a, b]$ is an expression of the form

$$\sum_{i=1}^{n} f(c_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_{i-1} \le c_i \le x_i$$

for each $i = 1, 2, \ldots, n$.

If the limit of these sums exists as $n \to \infty$, then we define the *definite integral* by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

## 3 Standard Choices of Sample Points

The left endpoint, right endpoint, and midpoint rules are all special cases of the Riemann sum definition.

If we choose

$$c_i = x_{i-1},$$

then we obtain the left endpoint rule:

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

If we choose

$$c_i = x_i,$$

then we obtain the right endpoint rule:

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

If we choose

$$c_i = \frac{x_{i-1} + x_i}{2},$$

then we obtain the midpoint rule:

$$\sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x.$$

Thus, a Riemann sum is simply a rectangle approximation in which we are free to choose any sample point $c_i$ inside each subinterval.

## 4 A Telescoping Example for $\int_a^b e^x dx$

We now consider an example in which a clever choice of sample points leads to a telescoping sum.

Let

$$f(x) = e^x$$

on the interval $[a, b]$. As before, divide the interval into $n$ equal parts, so that

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x.$$

For each subinterval $[x_{i-1}, x_i]$, define

$$c_i = x_{i-1} + \ln\left(\frac{e^{\Delta x} - 1}{\Delta x}\right).$$

We first verify that $c_i$ really does lie in the interval $[x_{i-1}, x_i]$.

**Claim**

For each $i$,

$$x_{i-1} \le c_i \le x_i.$$

To prove the left inequality, it is enough to show that

$$\ln\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) \ge 0.$$

Since the natural logarithm is increasing, this is equivalent to

$$\frac{e^{\Delta x} - 1}{\Delta x} \ge 1,$$

or

$$e^{\Delta x} - 1 \ge \Delta x.$$

This is true because

$$e^t \ge 1 + t$$

for all $t$, so in particular

$$e^{\Delta x} \ge 1 + \Delta x.$$

To prove the right inequality, it is enough to show that

$$\ln\left(\frac{e^{\Delta x} - 1}{\Delta x}\right) \le \Delta x.$$

Again using that the natural logarithm is increasing, this is equivalent to

$$\frac{e^{\Delta x} - 1}{\Delta x} \le e^{\Delta x}.$$

Multiplying by $\Delta x > 0$, we obtain

$$e^{\Delta x} - 1 \le \Delta x e^{\Delta x}.$$

This inequality is true because

$$e^{\Delta x} - e^0 = \int_0^{\Delta x} e^t \, dt \le \int_0^{\Delta x} e^{\Delta x} \, dt = \Delta x e^{\Delta x}.$$

Therefore,

$$x_{i-1} \le c_i \le x_i.$$

Thus, the numbers $c_i$ are valid sample points for a Riemann sum.

**Computation of the Sum**

Now compute

$$\begin{aligned} e^{c_i}\Delta x &= e^{x_{i-1}+\ln\left(\frac{e^{\Delta x}-1}{\Delta x}\right)}\Delta x \\ &= e^{x_{i-1}}\left(\frac{e^{\Delta x}-1}{\Delta x}\right)\Delta x \\ &= e^{x_{i-1}}(e^{\Delta x}-1). \end{aligned}$$

Since

$$x_i = x_{i-1} + \Delta x,$$

we have

$$e^{x_{i-1}}e^{\Delta x} = e^{x_i}.$$

Therefore,

$$e^{c_i}\Delta x = e^{x_i} - e^{x_{i-1}}.$$

Now form the corresponding Riemann sum:

$$\begin{aligned} \sum_{i=1}^{n} e^{c_i}\Delta x &= \sum_{i=1}^{n}\left(e^{x_i} - e^{x_{i-1}}\right) \\ &= \left(e^{x_1} - e^{x_0}\right) + \left(e^{x_2} - e^{x_1}\right) + \cdots + \left(e^{x_n} - e^{x_{n-1}}\right). \end{aligned}$$

This is a telescoping series, so all of the middle terms cancel. Hence,

$$\sum_{i=1}^{n} e^{c_i}\Delta x = e^{x_n} - e^{x_0}.$$

Since

$$x_n = b \qquad \text{and} \qquad x_0 = a,$$

we obtain

$$\sum_{i=1}^{n} e^{c_i}\Delta x = e^b - e^a.$$

This equality holds for every $n$, so taking the limit gives

$$\int_a^b e^x\,dx = e^b - e^a.$$

## 5 Summary

A Riemann sum has the form

$$\sum_{i=1}^{n} f(c_i)\Delta x,$$

where each $c_i$ is chosen inside the corresponding subinterval $[x_{i-1}, x_i]$. The left endpoint, right endpoint, and midpoint rules are special cases of this construction.

The example above is important because it shows that some choices of sample points are especially convenient. For the function $f(x) = e^x$, the choice

$$c_i = x_{i-1} + \ln\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)$$

turns the Riemann sum into a telescoping series, which immediately gives

$$\int_a^b e^x \, dx = e^b - e^a.$$

In general, Riemann sums provide the bridge between area approximations and exact values of definite integrals.