Riemann Sums

Math 140: Calculus with Analytic Geometry

1 Riemann Sums and the Definite Integral

Definition 1.1 (Riemann sum). Let f be defined on $[a, b]$, where $a < b$. For a positive integer n, divide $[a, b]$ into n equal subintervals, with endpoints

$$a = x_0 < x_1 < \cdots < x_n = b \qquad \text{and width} \qquad \Delta x = \frac{b-a}{n}.$$

Choose any point c_i in $[x_{i-1}, x_i]$ for each $i = 1, \ldots, n$. The sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x$$

is a *Riemann sum* for f on $[a, b]$. Each term is the signed contribution of a rectangle of width Δx. Its height is $|f(c_i)|$, and its contribution is negative when $f(c_i) < 0$.

Key idea. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

The limit is the same for every choice of sample points, even if we choose them anew for each n.

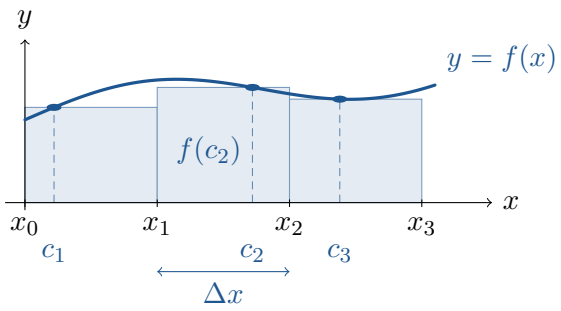


Figure 1: Three admissible sample points and their rectangles.

The sums A_n^{left}, A_n^{right}, and A_n^{midpoint} from *Approximation Sums* come from choosing the left endpoint, the right endpoint, or the midpoint in each subinterval. The next three examples use the freedom to choose c_i so that neighboring terms cancel.

2 Telescoping Riemann Sums

Example 2.1 (Any sample points for a constant function). Evaluate $\int_0^3 3\,dx$ using a Riemann sum.

Since every sample height is 3, any choice of c_i works. Writing $\Delta x = x_i - x_{i-1}$ makes the cancellation visible. With three subintervals, for instance,

$$3(x_1 - x_0) + 3(x_2 - x_1) + 3(x_3 - x_2) = 3(x_3 - x_0).$$

Each interior endpoint appears with opposite signs. A sum that cancels in this way is called a *telescoping sum*. For any n,

$$\sum_{i=1}^{n} 3\,\Delta x = \sum_{i=1}^{n} 3(x_i - x_{i-1}) = 3(x_n - x_0) = 9.$$

The sum is 9 for every n, so $\int_0^3 3\,dx = 9$.

Example 2.2 (Midpoints for a linear function). Evaluate $\int_0^3 2x\,dx$ using a Riemann sum.

Choose the midpoint $c_i = (x_{i-1} + x_i)/2$ of each subinterval. The sampled height is $2c_i = x_i + x_{i-1}$. Multiplying by the width $\Delta x = x_i - x_{i-1}$ gives a difference of squares.

$$2c_i\,\Delta x = (x_i + x_{i-1})(x_i - x_{i-1}) = x_i^2 - x_{i-1}^2.$$

Now add the contributions. Each interior square occurs with opposite signs in consecutive terms, so

$$\sum_{i=1}^{n} 2c_i\,\Delta x = x_n^2 - x_0^2 = 3^2 - 0^2 = 9.$$

The sum equals 9 for every n, so its limit is 9. Therefore $\int_0^3 2x\,dx = 9$.

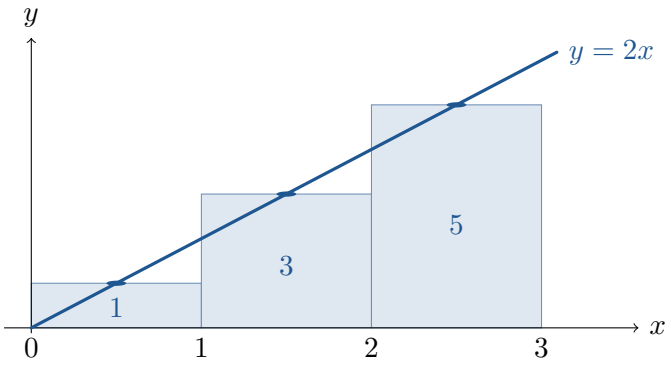


Figure 2: The midpoint rectangles for $n = 3$ have contributions 1, 3, and 5.

Example 2.3 (Choosing points for a quadratic function). Evaluate $\int_0^2 x^2\,dx$ using a Riemann sum.

For $2x$, factoring a difference of squares exposed the width Δx. For x^2, try the same idea with a difference of cubes. Since $x_i - x_{i-1} = \Delta x$,

$$\frac{x_i^3 - x_{i-1}^3}{3} = \Delta x \left(\frac{x_{i-1}^2 + x_{i-1}x_i + x_i^2}{3} \right).$$

To make $c_i^2\,\Delta x$ equal this difference, choose

$$c_i = \sqrt{\frac{x_{i-1}^2 + x_{i-1}x_i + x_i^2}{3}}.$$

To check that c_i lies in the subinterval, note that $0 \le x_{i-1} < x_i$, so $x_{i-1}^2 \le x_{i-1}x_i \le x_i^2$. The average under the square root lies strictly between x_{i-1}^2 and x_i^2, so $x_{i-1} < c_i < x_i$.
When we add the contributions, each interior cube appears with opposite signs. Thus

$$\sum_{i=1}^{n} c_i^2\,\Delta x = \sum_{i=1}^{n} \frac{x_i^3 - x_{i-1}^3}{3} = \frac{2^3 - 0^3}{3} = \frac{8}{3}.$$

The sum is $8/3$ for every n, so $\int_0^2 x^2\,dx = \frac{8}{3}$. The sample points matter for a fixed n. For example, the midpoint sum for $n = 1$ is $2 \cdot 1^2 = 2$.

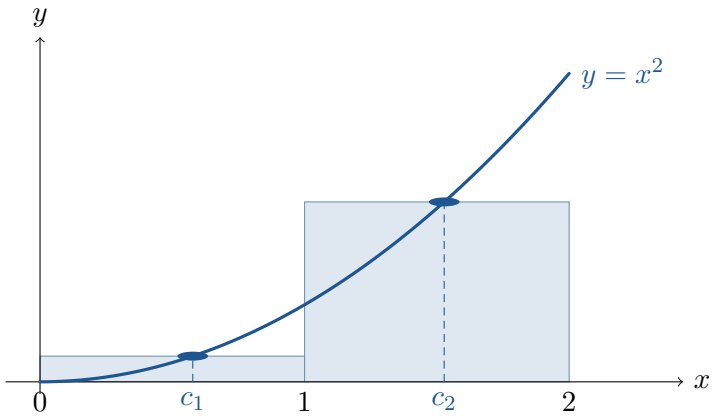


Figure 3: For $n = 2$, the widths are 1 and the heights are $1/3$ and $7/3$.

3 Why a Telescoping Choice Exists

In all three examples, the contribution on $[x_{i-1}, x_i]$ was the difference between the values of the same expression at the right and left endpoints. The Mean Value Theorem explains how to get this difference whenever we have a suitable antiderivative.

Recall the Mean Value Theorem. If F is continuous on a closed interval and differentiable on its interior, then at some interior point the tangent slope equals the secant slope across the interval.

Definition 3.1 (Antiderivative). A function F is an *antiderivative* of f on an open interval if $F'(x) = f(x)$ at every point of that interval.

Theorem 3.1 (Fundamental Theorem of Calculus, evaluation formula). Suppose f is continuous on $[a, b]$ and F is continuous on $[a, b]$ with $F' = f$ on (a, b). Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

To prove the formula, divide $[a, b]$ into n equal subintervals. On each $[x_{i-1}, x_i]$, F is continuous and is differentiable in the interior. The Mean Value Theorem therefore gives a point c_i strictly between the endpoints such that

$$f(c_i) = F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Since the denominator is Δx, the rectangle contribution is

$$f(c_i)\,\Delta x = F(x_i) - F(x_{i-1}).$$

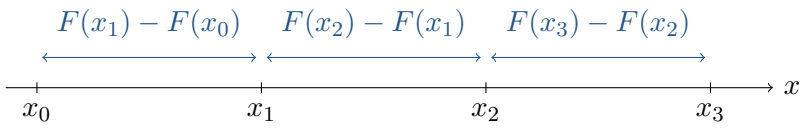


Figure 4: Each interior endpoint value appears with both signs.

Adding the contributions cancels $F(x_i)$ at every interior endpoint. For every n, the chosen Riemann sum therefore satisfies

$$\sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{n} \Big(F(x_i) - F(x_{i-1})\Big) = F(b) - F(a).$$

Since f is continuous, the limit of these Riemann sums is the definite integral. Every sum equals the same endpoint difference, so

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = F(b) - F(a).$$