

Rolle's Theorem and the Mean Value Theorem

Math 140: Calculus with Analytic Geometry

Key Topics

- Rolle's Theorem
- The Mean Value Theorem
- Geometric interpretation of derivatives
- Uniqueness of roots
- Inequalities and bounds
- Cauchy's Mean Value Theorem

1 Rolle's Theorem

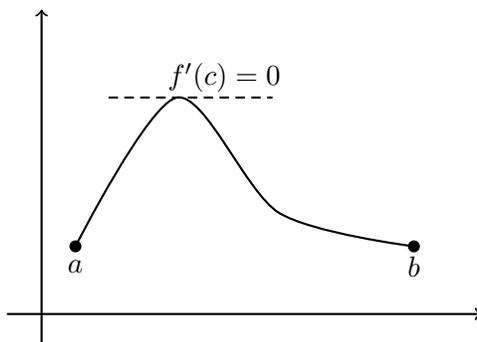
Theorem 1.1 (Rolle's Theorem)

Let f satisfy:

- f is continuous on $[a, b]$,
- f is differentiable on (a, b) ,
- $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$



Geometric Meaning

If a function begins and ends at the same height, then somewhere in between the tangent line must be horizontal.

Example 1.1 (Uniqueness of Roots)

Show that

$$f(x) = x^3 - 3x + 1$$

has at most one root in the interval $[0, 1]$.

Solution.

First compute the derivative:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

On the interval $(0, 1)$, we have $x^2 < 1$, so

$$f'(x) < 0.$$

Thus f is strictly decreasing on $(0, 1)$.

Now suppose, for contradiction, that f had two distinct roots $r_1 < r_2$ in $[0, 1]$. Then $f(r_1) = f(r_2) = 0$, so by Rolle's Theorem there would exist $c \in (r_1, r_2)$ such that $f'(c) = 0$.

But we have shown $f'(x) < 0$ on $(0, 1)$, so no such c exists.

Therefore, f has at most one root in $[0, 1]$.

Important Principle.

If $f'(x)$ does not change sign on an interval, then f is monotonic there and can have at most one root.

2 The Mean Value Theorem

Theorem 2.1 (Mean Value Theorem)

Let f be continuous on $[a, b]$ and differentiable on (a, b) .

Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometric Interpretation

The Mean Value Theorem says that the slope of the secant line over $[a, b]$ equals the slope of the tangent line at some interior point.

To make this completely precise, consider

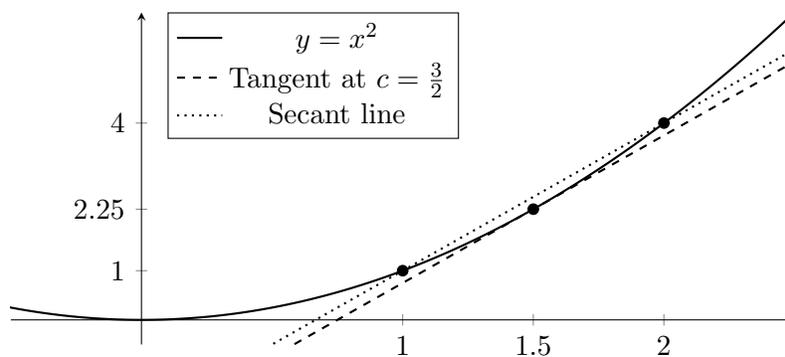
$$f(x) = x^2 \quad \text{on } [1, 2].$$

The secant slope is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{1} = 3.$$

Since $f'(x) = 2x$, we solve $2c = 3$ and obtain $c = \frac{3}{2}$.

Thus, the tangent line at $x = \frac{3}{2}$ has slope 3, so it is parallel to the secant line.



3 Applications of the Mean Value Theorem

Application 1: Monotonicity

If $f'(x) > 0$ on (a, b) , then f is increasing on $[a, b]$.

Reason. For $x_1 < x_2$, the MVT gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c \in (x_1, x_2)$. Since $f'(c) > 0$ and $x_2 - x_1 > 0$, the difference is positive.

If $f'(x) < 0$ on (a, b) , then f is decreasing on $[a, b]$.

Reason. For $x_1 < x_2$, the MVT gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c \in (x_1, x_2)$. Since $f'(c) < 0$ and $x_2 - x_1 > 0$, the difference is negative.

Application 2: Bounds for Exponential Function

Prove that for all $x \in (-1, 1)$

$$1 + x \leq e^x \leq \frac{1}{1 - x}.$$

Let $f(t) = e^t$ and $g(t) = e^{-t}$. Let $x \in (0, 1)$. Then, applying MVT to $f(t)$ on $[0, x]$ gives

$$e^x - e^0 = e^c(x - 0)$$

for some $c \in (0, x)$. Since $e^c \geq 1$, $e^x \geq 1 + x$. Applying MVT to $g(t)$ on $[0, x]$ gives

$$e^{-x} - e^0 = -e^{-c}(x - 0)$$

for some $c \in (0, x)$. Since $e^{-c} \leq 1$, $e^{-x} \geq 1 - x$. Therefore, $e^x \leq \frac{1}{1-x}$.

Let $x \in (-1, 0)$. Then, applying MVT to $f(t)$ on $[x, 0]$ gives

$$e^0 - e^x = e^c(0 - x)$$

for some $c \in (x, 0)$. Since $e^c \leq 1$, it follows that $1 - e^x \leq -x$, that is, $e^x \geq 1 + x$. Applying MVT to $g(t)$ on $[x, 0]$ gives

$$e^0 - e^{-x} = -e^{-c}(0 - x) = xe^{-c}$$

for some $c \in (x, 0)$. Since $e^{-c} \geq 1$ and $x < 0$, it follows that $1 - e^{-x} \leq x$. Therefore, $e^x \leq \frac{1}{1-x}$.

Application 3: Special Limits

Note that each $x \in (-1, 1)$, we have

$$1 + x \leq e^x \leq \frac{1}{1 - x}.$$

Therefore, for each $x \in (-1, 1)$,

$$x \leq e^x - 1 \leq \frac{x}{1 - x}.$$

Assuming $x > 0$, we have

$$1 \leq \frac{e^x - 1}{x} \leq \frac{1}{1 - x}.$$

Taking the limit as $x \rightarrow 0$ gives

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

4 Why This Matters

Rolle's Theorem and the Mean Value Theorem:

- Connect local behavior (derivatives) to global behavior.
- Guarantee monotonicity and uniqueness of roots.
- Justify powerful inequalities.
- Justify the fundamental theorem of calculus.