The Second Derivative Test

Math 140: Calculus with Analytic Geometry

Key Topics

- Concavity

- Inflection points

- The Second Derivative Test

- When the test fails

1 Concavity

Definition 1.1

Let f be twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then f is **concave up** on I.

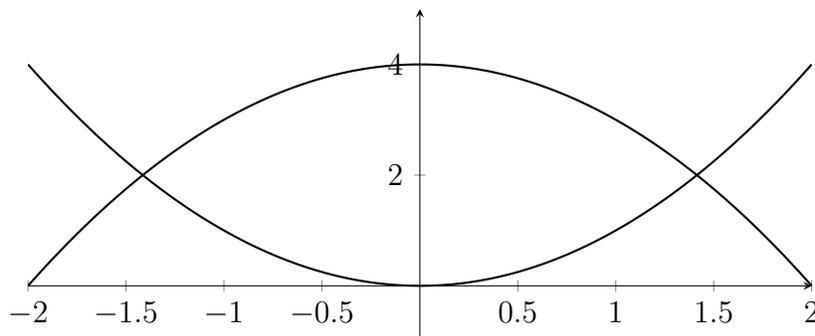- If $f''(x) < 0$ on I, then f is **concave down** on I.

Concavity describes how the slope is changing.

$$f''(x) > 0 \Rightarrow f'(x) \text{ increasing}$$
$$f''(x) < 0 \Rightarrow f'(x) \text{ decreasing}$$

Geometric Meaning

Note that $y = x^2$ is concave up and $y = -x^2 + 4$ is concave down.

2 Inflection Points

Definition 2.1

A point c is an **inflection point** if the concavity of f changes at c.
   Necessary condition (not sufficient):

$$f''(c) = 0 \quad \text{or} \quad f''(c) \text{ undefined.}$$

Example 2.1

$$f(x) = x^3.$$

$$f''(x) = 6x.$$

At $x = 0$, $f''(0) = 0$ and concavity changes sign.
Thus, $(0,0)$ is an inflection point.

3 The Second Derivative Test

Theorem 3.1 (Second Derivative Test)

Suppose:

$$f'(c) = 0 \quad \text{and} \quad f''(c) \text{ exists.}$$

Then:

- If $f''(c) > 0$, then f has a local minimum at c.

- If $f''(c) < 0$, then f has a local maximum at c.

- If $f''(c) = 0$, the test is inconclusive.

Why the Test Works

If $f'(c) = 0$, then c is a horizontal tangent.
If additionally:

- $f''(c) > 0$, then slopes are increasing near c. The graph bends upward, which implies a local minimum.

- $f''(c) < 0$, then slopes are decreasing near c. The graph bends downward, which implies a local maximum.

Thus the second derivative determines the curvature at the critical point.

4 Examples

Example 4.1

Classify the critical points of

$$f(x) = x^3 - 3x^2 + 2.$$

Solution.

$$f'(x) = 3x^2 - 6x = 3x(x - 2).$$

Critical numbers:
$$x = 0, \quad x = 2.$$

Second derivative:

$$f''(x) = 6x - 6.$$

Evaluate:

$$f''(0) = -6 < 0 \Rightarrow \text{local maximum at } x = 0.$$

$$f''(2) = 6 > 0 \Rightarrow \text{local minimum at } x = 2.$$

Example 4.2 (Test Fails)

$$f(x) = x^4.$$

$$f'(x) = 4x^3, \quad f''(x) = 12x^2.$$

Critical number: $x = 0$.

$$f''(0) = 0.$$

The test is inconclusive.

However, examining the graph or using the First Derivative Test shows that $x = 0$ is a local minimum.

5   Comparison with First Derivative Test

- The First Derivative Test uses sign changes of f'.

- The Second Derivative Test uses the sign of f''.

- The Second Derivative Test is faster but may fail when $f''(c) = 0$.

6   Why This Matters

The Second Derivative Test:

- Provides a quick classification tool.

- Connects curvature and extrema.

- Prepares us for optimization problems.

- Deepens understanding of how the derivative behaves.

Together with the First Derivative Test, it forms the foundation for analyzing the shape of graphs.