Tangent and Secant Lines

Math 140: Calculus with Analytic Geometry

Key Topics

- Secant lines and average rate of change
- Tangent lines as limits of secant lines
- The difference quotient
- Visualizing secants approaching a tangent as $h \to 0$

1 Secant Lines and Average Rate of Change

Definition 1.1. *Let f be a function and let $x = a$ and $x = b$ be two* distinct *points in the domain of f. The* ***secant line*** *to the graph of f through these points is the line passing through*

$$(a, f(a)) \quad \textit{and} \quad (b, f(b)).$$

The slope of this secant line is

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a}.$$

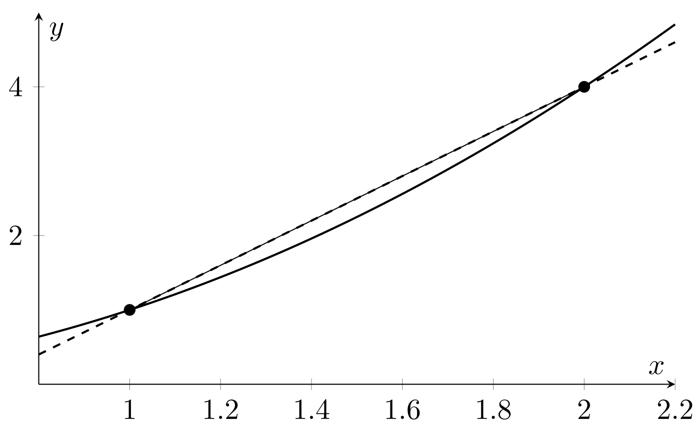
This slope is often interpreted as the **average rate of change** of f on the interval $[a, b]$.

1.1 Examples of Secant Lines

Example 1.1. *Let $f(x) = x^2$ on $[1, 2]$. The secant slope is*

$$m_{sec} = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{1} = 3.$$

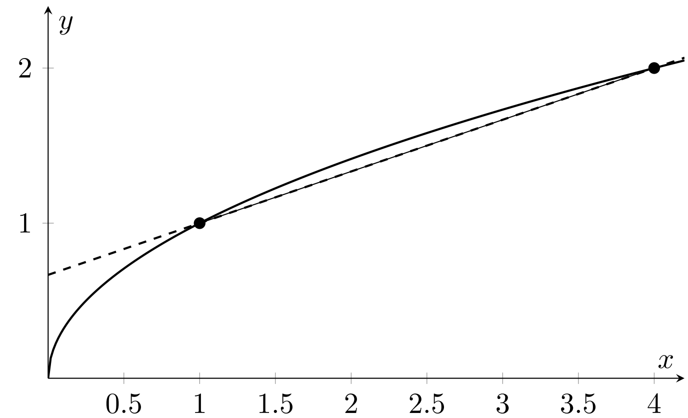
The secant line through $(1, 1)$ and $(2, 4)$ is $y - 1 = 3(x - 1)$, i.e. $y = 3x - 2$.



Example 1.2. *Let $f(x) = \sqrt{x}$ on $[1, 4]$. The secant slope is*

$$m_{sec} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$

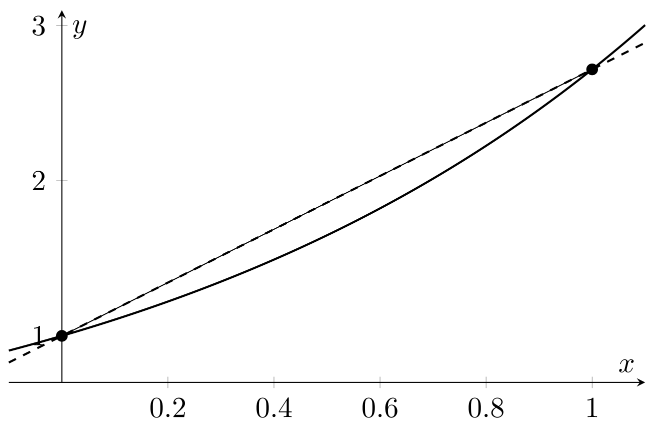
The secant line through $(1, 1)$ and $(4, 2)$ is $y - 1 = \frac{1}{3}(x - 1)$, i.e. $y = \frac{x}{3} + \frac{2}{3}$.



Example 1.3. *Let $f(x) = e^x$ on $[0, 1]$. The secant slope is*

$$m_{sec} = \frac{f(1) - f(0)}{1 - 0} = e - 1.$$

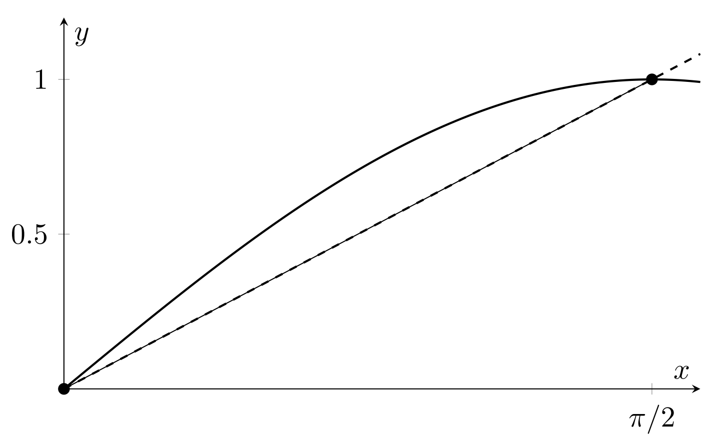
The secant line through $(0, 1)$ and $(1, e)$ is $y - 1 = (e - 1)x$, i.e. $y = (e - 1)x + 1$.



Example 1.4. *Let $f(x) = \sin(x)$ on $\left[0, \frac{\pi}{2}\right]$. The secant slope is*

$$m_{sec} = \frac{1 - 0}{\pi/2} = \frac{2}{\pi}.$$

The secant line is $y = \frac{2}{\pi}x$.



2 Tangent Lines

To move from an average rate of change to an instantaneous rate of change at a point, we let the second point on the secant line approach the first point.

2.1 The Difference Quotient

Writing the second point as $a + h$ gives the difference quotient

$$\frac{f(a+h) - f(a)}{h}.$$

2.2 Tangent Line as a Limit of Secants

Definition 2.1. *If the limit*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

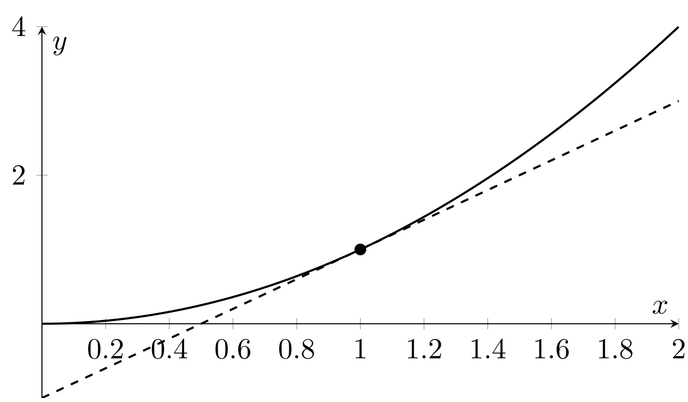
exists, it is the slope of the tangent line to $y = f(x)$ at $x = a$.

2.3 Examples of Tangent Lines

Example 2.1. *For $f(x) = x^2$, the tangent line at $(1,1)$ has slope*

$$\begin{aligned} m &= \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} \frac{2+h}{=} 2. \end{aligned}$$

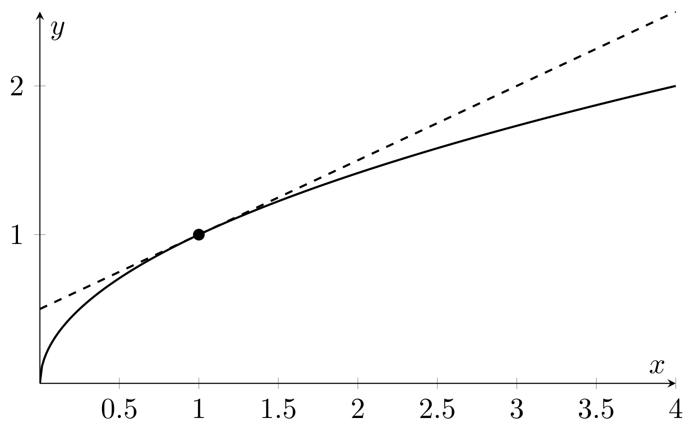
Therefore, the equation of the tangent line at $(1,1)$ is $y - 1 = 2(x - 1)$.



Example 2.2. *For $f(x) = \sqrt{x}$, the tangent line at $(1, 1)$ has slope*

$$\begin{aligned} m &= \lim_{h \to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\ &= \lim_{h \to 0} \frac{(1+h) - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2}. \end{aligned}$$

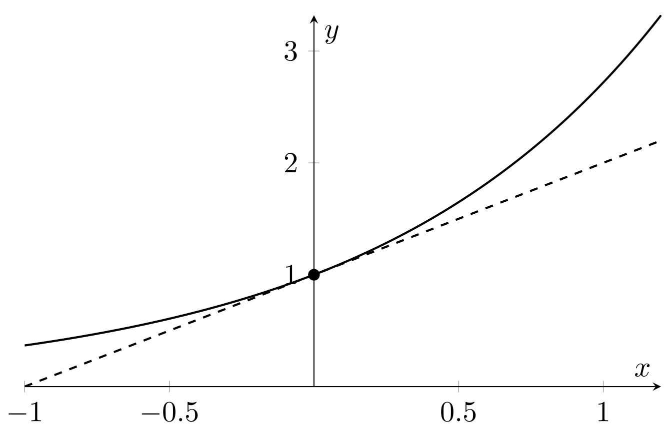
Hence, the equation of the tangent line at $(1, 1)$ is $y - 1 = \frac{1}{2}(x - 1)$.



Example 2.3. *For $f(x) = e^x$, the tangent line at $(0, 1)$ has slope*

$$\begin{aligned} m &= \lim_{h \to 0} \frac{e^{(0+h)} - e^0}{h} \\ &= \lim_{h \to 0} \frac{e^h - 1}{h} = 1. \end{aligned}$$

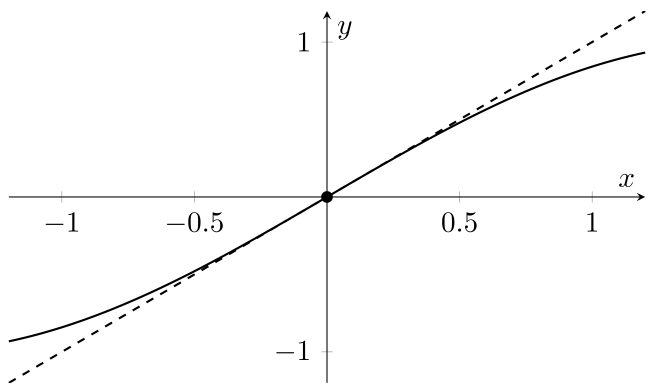
Hence, the equation of the tangent line at $(0, 1)$ is $y - 1 = 1(x - 0)$.



Example 2.4. *For $f(x) = \sin(x)$, the tangent line at $(0,0)$ has slope*

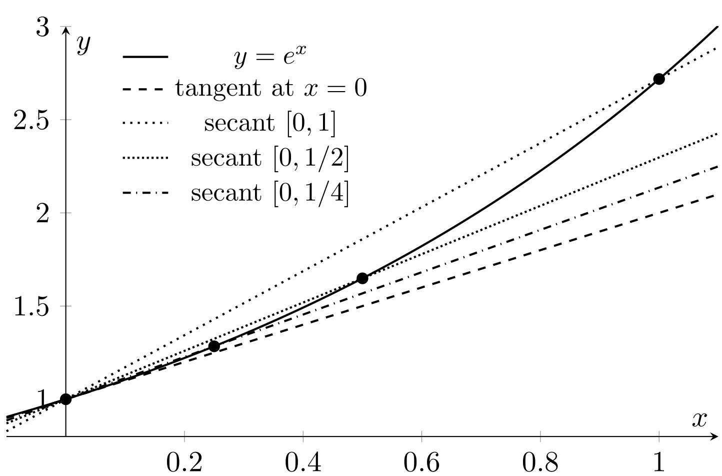
$$m = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h}$$
$$= \lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

Hence, the equation of the tangent line at $(0,0)$ is $y - 0 = 1(x - 0)$.



2.4 Secant Lines Approaching the Tangent Line

We now illustrate how secant lines for $f(x) = e^x$ approach the tangent line at $x = 0$. Consider the secant lines on the intervals $[0,1]$, $[0, \frac{1}{2}]$, and $[0, \frac{1}{4}]$.



3 Why This Matters for Calculus

- Tangent lines capture instantaneous change, which is central to velocity and growth.
- Secant lines provide computable approximations that lead naturally to limits.
- This geometric viewpoint motivates the formal definition of the derivative.