Trigonometric Functions

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Math 140: Calculus with Analytic Geometry

1 Learning Goals

After studying these notes, you should be able to:

- interpret angles in degrees and radians and use the arc-length formula;
- define all six trigonometric functions using right triangles and the unit circle;
- evaluate trigonometric functions exactly at common angles;
- use symmetry, periodicity, reference angles, and fundamental identities;
- identify the important features and transformations of trigonometric graphs;
- solve basic trigonometric equations; and
- evaluate inverse trigonometric functions and distinguish an inverse from a reciprocal.

Throughout calculus, angles are measured in *radians* unless a degree symbol is shown. This convention matters: the standard derivative formulas for sine and cosine are valid in their familiar form only when the input is measured in radians.

2 Angles and Radian Measure

An angle is in *standard position* when its vertex is at the origin and its initial side lies on the positive horizontal axis. Counterclockwise rotation gives a positive angle, and clockwise rotation gives a negative angle. Angles that differ by a whole number of complete revolutions are *coterminal*; they have the same terminal side.

Definition 2.1. For a circle of radius r, an angle subtending an arc of length s has radian measure

$$\theta = \frac{s}{r}.$$

Equivalently, the length of the intercepted arc is

$$s = r\theta,$$

provided that θ is measured in radians.

Since the circumference of a circle is $2\pi r$, one complete revolution is 2π radians. Thus

$$2\pi \text{ radians} = 360^\circ, \qquad \pi \text{ radians} = 180^\circ.$$

The conversion rules are therefore

$$\text{degrees} \cdot \frac{\pi}{180} = \text{radians}, \qquad \text{radians} \cdot \frac{180}{\pi} = \text{degrees}.$$

Example 2.1. Convert 225° to radians and $-5\pi/6$ to degrees.

$$225^\circ \left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{4}, \qquad -\frac{5\pi}{6}\left(\frac{180^\circ}{\pi}\right) = -150^\circ.$$

Example 2.2. A wheel of radius 10 centimeters turns through $3\pi/5$ radians. The point on its rim travels

$$s = r\theta = 10\left(\frac{3\pi}{5}\right) = 6\pi \text{ centimeters}.$$

Remark 2.1. Radian measure is dimensionless: it is the ratio of two lengths. On the unit circle, $r = 1$, so the numerical value of θ equals the length of the intercepted arc.

3 Right-Triangle Trigonometry

Let θ be an acute angle in a right triangle. Relative to θ, call the side across from the angle the *opposite* side, the nonhypotenuse side touching the angle the *adjacent* side, and the longest side the *hypotenuse*.

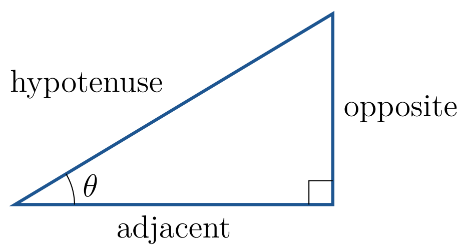

Figure 1: Side names are determined relative to the chosen acute angle.

Definition 3.1. The six trigonometric ratios are

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}},$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}, \quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}, \quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

The last three functions are reciprocals of the first three:

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Also, $\tan\theta = \sin\theta/\cos\theta$ and $\cot\theta = \cos\theta/\sin\theta$ wherever the quotients are defined.

Example 3.1. Suppose the side opposite θ has length 5 and the adjacent side has length 12. The Pythagorean theorem gives a hypotenuse of 13. Hence

$$\sin\theta = \frac{5}{13}, \quad \cos\theta = \frac{12}{13}, \quad \tan\theta = \frac{5}{12}, \quad \csc\theta = \frac{13}{5}, \quad \sec\theta = \frac{13}{12}, \quad \cot\theta = \frac{12}{5}.$$

Example 3.2. Let x be an acute angle in a right triangle and suppose that

$$\cos x = \frac{2}{3}.$$

This is the right-triangle trigonometry problem from the precalculus review quiz. Since cosine is adjacent over hypotenuse, draw a right triangle with adjacent side 2 and hypotenuse 3. If the opposite side has length a, then

$$a^2 + 2^2 = 3^2, \qquad a^2 = 5, \qquad a = \sqrt{5}.$$

The positive root is used because a side length is positive. It follows that

$$\sin x = \frac{\sqrt{5}}{3}, \qquad \tan x = \frac{\sqrt{5}}{2}.$$

This example illustrates a general strategy: translate the given ratio into two side lengths, use the Pythagorean theorem to find the third side, and then form the requested ratios.

4 The Unit Circle

The right-triangle definitions apply directly only to acute angles. The unit circle extends the definitions to every real input.

Definition 4.1. The *unit circle* is the circle $x^2 + y^2 = 1$. Starting at $(1, 0)$, sweep out an angle θ in standard position. The terminal side of θ intersects the unit circle at a point $P = (x, y)$, where

$$\cos\theta = x, \qquad \sin\theta = y,$$

see Figure 2. Whenever the relevant denominator is nonzero,

$$\tan\theta = \frac{y}{x}, \qquad \sec\theta = \frac{1}{x}, \qquad \csc\theta = \frac{1}{y}, \qquad \cot\theta = \frac{x}{y}.$$

The 30–60–90 and 45–45–90 triangles supply all first-quadrant special values. Signs are then determined by the quadrant.

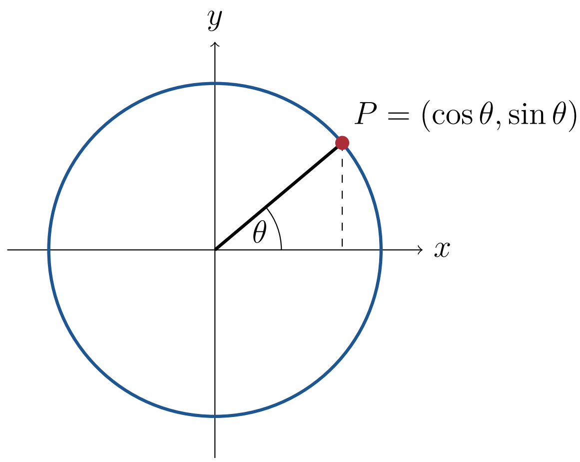

Figure 2: The coordinate definition of sine and cosine on the unit circle.

The exact first-quadrant values can be recorded accessibly as follows.

$\theta = 0$: $\sin\theta = 0$, $\cos\theta = 1$, and $\tan\theta = 0$.

$\theta = \pi/6$: $\sin\theta = 1/2$, $\cos\theta = \sqrt{3}/2$, and $\tan\theta = \sqrt{3}/3$.

$\theta = \pi/4$: $\sin\theta = \sqrt{2}/2$, $\cos\theta = \sqrt{2}/2$, and $\tan\theta = 1$.

$\theta = \pi/3$: $\sin\theta = \sqrt{3}/2$, $\cos\theta = 1/2$, and $\tan\theta = \sqrt{3}$.

$\theta = \pi/2$: $\sin\theta = 1$, $\cos\theta = 0$, and $\tan\theta$ is undefined.

4.1 Reference angles and signs

The *reference angle* α is the acute angle between the terminal side of θ and the horizontal axis. The magnitudes of the trigonometric values at θ match those at α; the quadrant determines the signs.

- Quadrant I: all six functions are positive.
- Quadrant II: sine and cosecant are positive.
- Quadrant III: tangent and cotangent are positive.
- Quadrant IV: cosine and secant are positive.

Example 4.1. Evaluate $\sin(5\pi/6)$, $\cos(5\pi/6)$, and $\tan(5\pi/6)$. The reference angle is $\pi/6$, and $5\pi/6$ lies in Quadrant II. Therefore

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}, \qquad \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \qquad \tan\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{3}.$$

Example 4.2. Evaluate $\cos(-7\pi/4)$. Adding 2π produces the coterminal angle $\pi/4$, so

$$\cos(-7\pi/4) = \cos(\pi/4) = \frac{\sqrt{2}}{2}.$$

4.2 Identities, Symmetry, and Periodicity

Because $(\cos\theta, \sin\theta)$ lies on $x^2 + y^2 = 1$,

$$\sin^2\theta + \cos^2\theta = 1.$$

Here $\sin^2\theta$ means $(\sin\theta)^2$. Dividing this identity by $\cos^2\theta$ or $\sin^2\theta$ gives the other Pythagorean identities:

$$1 + \tan^2\theta = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta.$$

Reflection across the horizontal axis shows that cosine is even and sine is odd:

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta.$$

Consequently, secant is even, while tangent, cotangent, and cosecant are odd.

One trip around the unit circle returns to the same point. Therefore sine and cosine have period 2π:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta.$$

Tangent and cotangent have period π because both numerator and denominator change sign after a half-turn:

$$\tan(\theta + \pi) = \tan\theta, \qquad \cot(\theta + \pi) = \cot\theta.$$

Example 4.3. If $\sin\theta = 3/5$ and θ lies in Quadrant II, find $\cos\theta$ and $\tan\theta$. The Pythagorean identity gives

$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}.$$

Since cosine is negative in Quadrant II, $\cos\theta = -4/5$. Hence

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}.$$

The quadrant information is essential when choosing the sign of the square root.

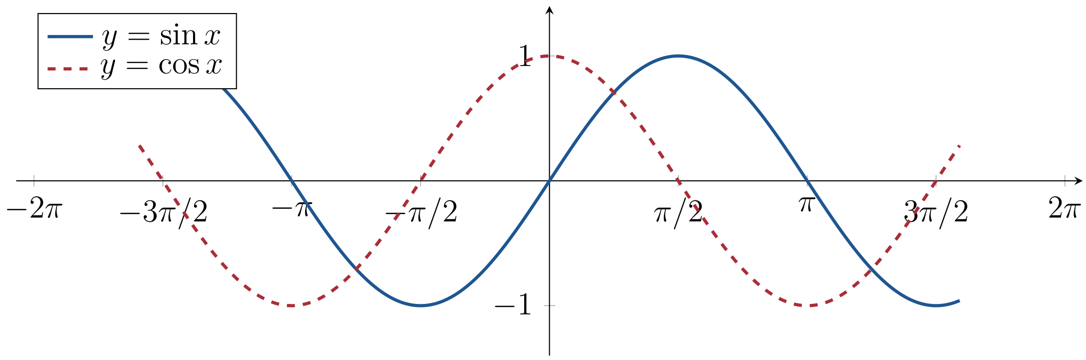

Figure 3: Sine and cosine over two periods.

5 Graphs of the Basic Trigonometric Functions

5.1 Sine and cosine

Both sine and cosine have domain $\mathbb{R}$, range $[-1, 1]$, and period 2π. Sine has zeros at $x = k\pi$, while cosine has zeros at $x = \pi/2 + k\pi$, where k is any integer.

For
$$y = A\sin(B(x - C)) + D \quad \text{or} \quad y = A\cos(B(x - C)) + D,$$
the amplitude is $|A|$, the period is $2\pi/|B|$, the horizontal shift is C, and the midline is $y = D$. The range is $[D - |A|, D + |A|]$. A negative A reflects the graph across its midline.

Example 5.1. For $f(x) = -2\cos(3(x - \pi/6)) + 1$, the amplitude is 2, the period is $2\pi/3$, the graph is shifted right by $\pi/6$, and its midline is $y = 1$. Its range is $[-1, 3]$.

5.2 Tangent

Since $\tan x = \sin x/\cos x$, tangent is undefined wherever $\cos x = 0$. Thus
$$\operatorname{dom}(\tan) = \left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\},$$
its range is $\mathbb{R}$, its zeros occur at $x = k\pi$, and its period is π.

For $y = A\tan(B(x - C)) + D$, the period is $\pi/|B|$, the horizontal shift is C, and the horizontal centerline is $y = D$. The word *amplitude* does not apply because tangent is unbounded.

5.3 Reciprocal functions

The reciprocal relationships explain the remaining graphs.

- $\sec x$ is undefined where $\cos x = 0$ and has range $(-\infty, -1] \cup [1, \infty)$.
- $\csc x$ is undefined where $\sin x = 0$ and has the same range as secant.
- $\cot x$ is undefined where $\sin x = 0$, has range $\mathbb{R}$, and has period π.

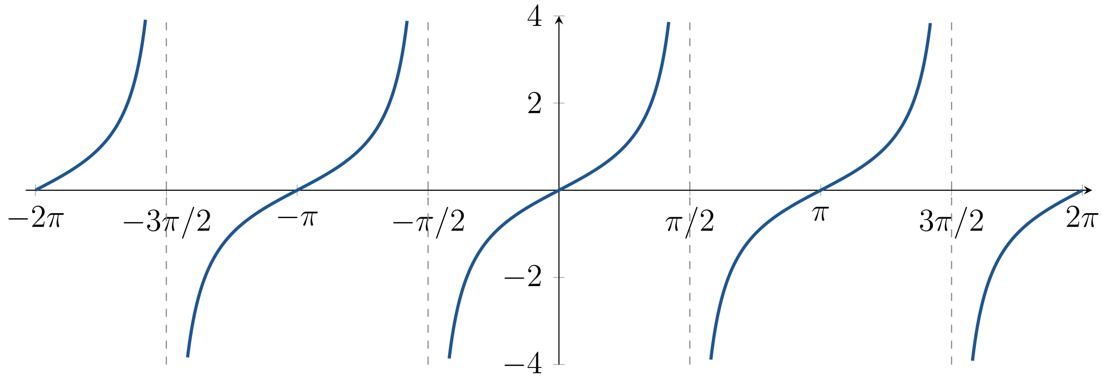

Figure 4: The tangent graph and its vertical asymptotes.

Zeros of a denominator become vertical asymptotes of the reciprocal function. The reciprocal functions can never equal zero.

6 Solving Basic Trigonometric Equations

An inverse trigonometric function usually returns only one *principal value*. A trigonometric equation may have many solutions because the original function is periodic. Use the unit circle to find all angles in one period, then use periodicity.

Example 6.1. Solve $\sin x = \sqrt{3}/2$ for all real x. In one revolution, sine has this value at $x = \pi/3$ and $x = 2\pi/3$. Therefore

$$x = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad x = \frac{2\pi}{3} + 2\pi k, \qquad k \in \mathbb{Z}.$$

Example 6.2. Solve $2\cos^2 x - 1 = 0$ on $[0, 2\pi)$. We obtain

$$\cos^2 x = \frac{1}{2}, \qquad \cos x = \pm\frac{\sqrt{2}}{2}.$$

The solutions in the requested interval are

$$x = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4}.$$

Taking only the positive square root would lose two solutions.

7 Inverse Trigonometric Functions

No periodic function is one-to-one on its entire domain. We obtain inverse functions by restricting sine, cosine, and tangent to intervals on which they pass the horizontal line test.

Definition 7.1. The principal inverse trigonometric functions are defined by the following restrictions.

- $\arcsin x$ is the inverse of sine restricted to $[-\pi/2, \pi/2]$. Its domain is $[-1, 1]$ and its range is $[-\pi/2, \pi/2]$.
- $\arccos x$ is the inverse of cosine restricted to $[0, \pi]$. Its domain is $[-1, 1]$ and its range is $[0, \pi]$.
- $\arctan x$ is the inverse of tangent restricted to $(-\pi/2, \pi/2)$. Its domain is $\mathbb{R}$ and its range is $(-\pi/2, \pi/2)$.

The notations $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ mean inverse functions, not reciprocals. For example,

$$\sin^{-1} x = \arcsin x, \qquad \frac{1}{\sin x} = \csc x.$$

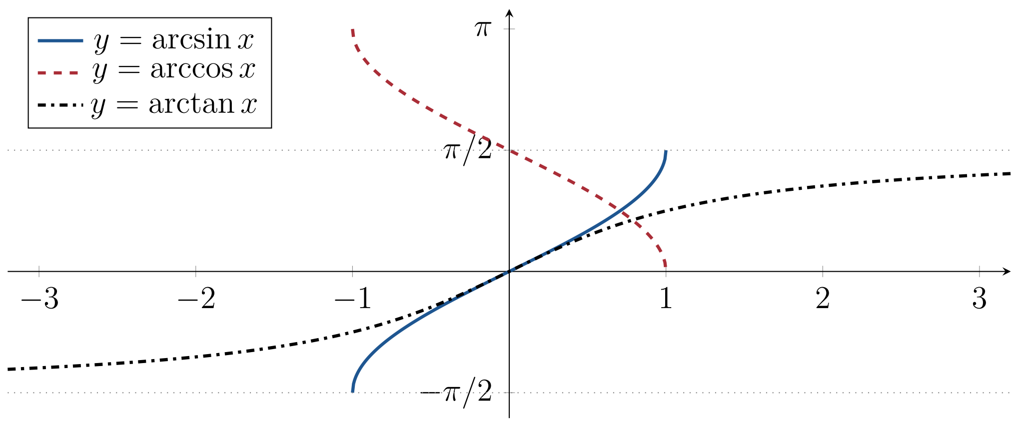

Figure 5: Graphs and principal-value ranges of the inverse trigonometric functions.

Example 7.1. Evaluate $\arcsin(-\sqrt{2}/2)$, $\arccos(-\sqrt{2}/2)$, and $\arctan(-1)$. Each answer must lie in the range of its inverse function:

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}, \qquad \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}, \qquad \arctan(-1) = -\frac{\pi}{4}.$$

Inverse-function cancellation depends on the input:

$$\begin{aligned} \sin(\arcsin x) &= x \quad \text{for } -1 \le x \le 1, \\ \cos(\arccos x) &= x \quad \text{for } -1 \le x \le 1, \\ \tan(\arctan x) &= x \quad \text{for every real } x. \end{aligned}$$

In the other order, cancellation occurs only on the restricted interval. For example, $\arcsin(\sin x) = x$ only when $x \in [-\pi/2, \pi/2]$.

Example 7.2. Evaluate $\arcsin(\sin(5\pi/6))$. Since $5\pi/6$ is outside the range of arcsine, we cannot cancel. Instead,

$$\sin(5\pi/6) = \frac{1}{2}, \qquad \arcsin(1/2) = \frac{\pi}{6}.$$

Thus $\arcsin(\sin(5\pi/6)) = \pi/6$, not $5\pi/6$.

Example 7.3. Simplify $\cos(\arcsin x)$ for $-1 \le x \le 1$. Let $\theta = \arcsin x$. Then $\sin\theta = x$ and $\theta \in [-\pi/2, \pi/2]$, where cosine is nonnegative. The Pythagorean identity gives

$$\cos(\arcsin x) = \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}.$$

The positive square root is forced by the principal range of arcsine.

8 Why This Matters for Calculus

This review supports several central ideas that appear immediately in calculus:

- radian measure makes the limit $\lim_{x\to 0} \sin x/x = 1$ possible in its standard form;
- the identities and unit-circle signs are used when simplifying limits and derivatives;
- periodicity and boundedness are important in graphical and limit analysis;
- domains of tangent, secant, cosecant, and cotangent identify possible discontinuities;
- inverse trigonometric functions appear in derivative formulas and antiderivatives; and
- transformations of sinusoidal functions model periodic motion and oscillation.