U-Substitution

Math 140: Calculus with Analytic Geometry

1 Reversing the Chain Rule

The Fundamental Theorem of Calculus lets us evaluate a definite integral once we know an antiderivative. U-substitution helps us find one when the integrand contains a composition together with the derivative of its inner function. Recall that if $F' = f$, then the chain rule gives

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x).$$

Reading this identity in reverse gives the substitution rule.

Theorem 1.1 (Substitution for indefinite integrals). Suppose g is differentiable on an interval I and f is continuous on an interval containing the values of g on I. If $F' = f$, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

With $u = g(x)$ and $du = g'(x)\,dx$, we write the integral as $\int f(u)\,du$. After finding $F(u)$, we replace u with $g(x)$.

Example 1.1 (Matching the inner derivative). Evaluate $\int 2x(x^2+1)^3\,dx$.

$$u = x^2 + 1, \qquad du = 2x\,dx.$$

The power $(x^2+1)^3$ becomes u^3, and the entire factor $2x\,dx$ becomes du. Thus

$$\int 2x(x^2+1)^3\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

To check, differentiation first brings down a factor of 4 and then multiplies by the inner derivative $2x$:

$$\frac{d}{dx}\left[\frac{(x^2+1)^4}{4}\right] = (x^2+1)^3(2x).$$

Key idea. After substitution, every factor in the integral should be expressed in u, with differential du. The notation $du = g'(x)\,dx$ records the factor supplied by the chain rule.

Which variable should the answer use?

An indefinite integral asks for a family of functions of x. Use u to find an antiderivative, then write $F(g(x)) + C$ so that differentiating with respect to x recovers the original integrand. A definite integral asks for a number. After converting both bounds, evaluate the antiderivative at those u-values and subtract. Section 3 develops this procedure.

2 Choosing a Substitution

Look for an expression that is raised to a power, under a radical, in a denominator, or in an exponent. Differentiate that expression and compare its derivative with the remaining factors. A constant multiplier can be moved outside the integral.

Example 2.1 (A linear inner function). Evaluate $\int (3x-1)^5\,dx$. With $u = 3x-1$, we have $du = 3\,dx$, so

$$\int (3x-1)^5\,dx = \frac{1}{3}\int u^5\,du = \frac{(3x-1)^6}{18} + C.$$

The factor $1/3$ compensates for the factor of 3 that the chain rule introduces when we differentiate the answer.

Example 2.2 (A radical). Evaluate $\int x\sqrt{x^2+4}\,dx$. With $u = x^2+4$, we have $x\,dx = \frac{1}{2}\,du$. Since $u > 0$, write $\sqrt{u} = u^{1/2}$ and apply the power rule:

$$\int x\sqrt{x^2+4}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}(x^2+4)^{3/2} + C.$$

Example 2.3 (An exponential). Evaluate $\int xe^{-x^2}\,dx$. With $u = -x^2$, we have $du = -2x\,dx$. The negative sign carries through:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^{-x^2} + C.$$

Example 2.4 (Rewriting a remaining factor). Evaluate $\int x^3(x^2+1)^2\,dx$. Let $u = x^2+1$. Write $x^3\,dx = x^2(x\,dx)$. The two factors can be rewritten as $x^2 = u-1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x^3(x^2+1)^2\,dx = \frac{1}{2}\int (u-1)u^2\,du = \frac{(x^2+1)^4}{8} - \frac{(x^2+1)^3}{6} + C.$$

Example 2.5 (A negative power). Evaluate $\displaystyle\int \frac{x}{(x^2+4)^2}\,dx$. Let $u = x^2+4$, so $x\,dx = \frac{1}{2}\,du$. Write the denominator as a negative power and apply the power rule:

$$\int \frac{x}{(x^2+4)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2u} + C = -\frac{1}{2(x^2+4)} + C.$$

Key idea. For $\int x\sqrt{x^2+4}\,dx$, the factor x makes $u = x^2+4$ simplify the integral immediately. In $\int \sqrt{x^2+4}\,dx$, the same choice leaves dx to be rewritten. On $x > 0$ it gives $dx = du/(2\sqrt{u-4})$, so the new integrand still has two radicals. Check the whole integrand before committing to a substitution.

3 Changing the Bounds

The bounds of a definite integral mark endpoint values of its variable. When $u = g(x)$, the starting value $x = a$ gives $u = g(a)$ and the ending value $x = b$ gives $u = g(b)$.

Theorem 3.1 (Substitution for definite integrals). Let $a < b$. Suppose g has a continuous derivative on $[a, b]$ and f is continuous on an interval containing the values of g on $[a, b]$. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Take an antiderivative F of f. The chain rule gives $(F \circ g)'(x) = f(g(x))g'(x)$. By the Fundamental Theorem of Calculus,

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 3.1 (Changing both endpoints). Evaluate $\displaystyle\int_0^2 x(x^2+1)^3\,dx$. Let $u = x^2 + 1$, so $x\,dx = \frac{1}{2}\,du$. The bounds $x = 0$ and $x = 2$ give $u = 1$ and $u = 5$, respectively. Therefore

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\,\frac{u^4}{4}\bigg|_1^5 = \frac{5^4 - 1}{8} = 78.$$

Example 3.2 (Decreasing substitution). Evaluate $\displaystyle\int_0^1 x\sqrt{4-3x^2}\,dx$. With $u = 4 - 3x^2$, we have $x\,dx = -\frac{1}{6}\,du$. The starting value is $u = 4$ and the ending value is $u = 1$. Reversing the bounds contributes another minus sign:

$$\int_0^1 x\sqrt{4-3x^2}\,dx = -\frac{1}{6}\int_4^1 u^{1/2}\,du = \frac{1}{6}\int_1^4 u^{1/2}\,du = \frac{4^{3/2} - 1}{9} = \frac{7}{9}.$$

Example 3.3 (Equal transformed endpoints). Evaluate $\displaystyle\int_0^1 (1-2x)e^{x-x^2}\,dx$. Let $u = x - x^2$, so $du = (1 - 2x)\,dx$. Both original endpoints give $u = 0$, so the integral is $\int_0^0 e^u\,du = 0$. In between, u increases to $\frac{1}{4}$ and then decreases back to 0. Integrating along the same u-interval in opposite directions gives opposite signed contributions.

Key idea. For a definite integral, change the bounds and finish the evaluation in u. For an indefinite integral, substitute back and give the antiderivative in x.

4 Geometric Interpretation

A nonnegative integrand over an interval with increasing bounds gives an area under a graph. Once both integrals are in this form, their equality tells us that the two regions have the same area. The new region may be simple enough for a geometry formula.

Example 4.1 (From a cubic curve to a trapezoid). Find the area under $y = x(x^2 + 1)$ on $[0, 2]$. With $u = x^2 + 1$, the factor $x\,dx$ becomes $\frac{1}{2}\,du$. The new integrand is $u/2$, so the new region has a straight top boundary:

$$\int_0^2 x(x^2 + 1)\,dx = \int_1^5 \frac{u}{2}\,du.$$

The new region has width $5 - 1 = 4$ and vertical sides of lengths $\frac{1}{2}$ and $\frac{5}{2}$.

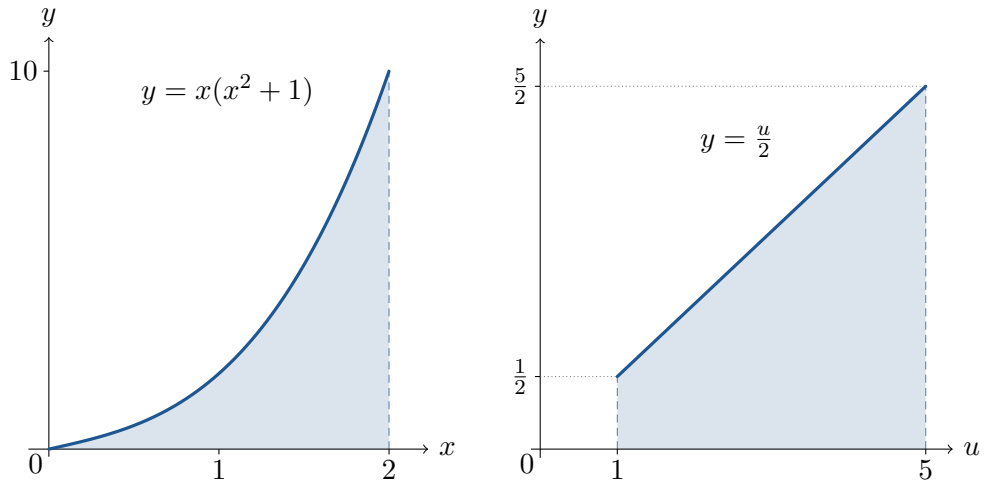


Figure 1: The region under the cubic and the trapezoid have equal area.

The area of a trapezoid is the distance between its parallel sides times the average of their lengths. Here that gives

$$A = 4\left(\frac{\frac{1}{2} + \frac{5}{2}}{2}\right) = 6.$$

We can also split the trapezoid into a rectangle of height $\frac{1}{2}$ and a triangle of height $\frac{5}{2} - \frac{1}{2} = 2$. They have the same base length 4, so their areas are 2 and 4, respectively.

Key idea. To read the new region from an integral, use the u-bounds to mark its horizontal interval and the transformed integrand to label its top boundary. The line $y = 0$ forms the bottom boundary in these examples.

4.1 A curve that becomes a rectangle

Example 4.2 (A decreasing substitution). Find the area under $y = x/\sqrt{4-x^2}$ on $[0, \sqrt{3}]$. The integrand is nonnegative on this interval. The radical itself is a convenient choice for u:

$$u = \sqrt{4-x^2}, \qquad du = -\frac{x}{\sqrt{4-x^2}}\,dx.$$

At $x = 0$ the new variable is $u = 2$. At $x = \sqrt{3}$ it is $u = 1$. The original left endpoint therefore becomes the right endpoint of the rectangle in the u-plane. Reversing the bounds absorbs the minus sign:

$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\,dx = -\int_2^1 1\,du = \int_1^2 1\,du.$$

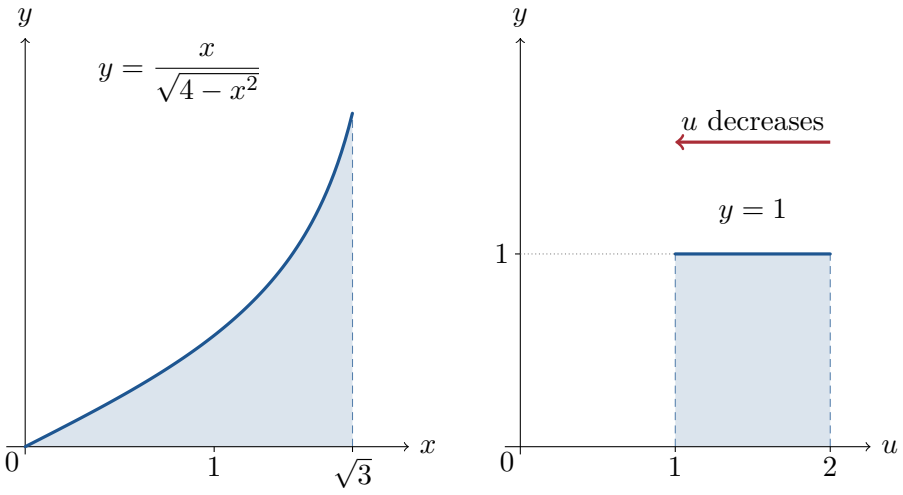


Figure 2: The curved region has the same area as a rectangle of height 1.

The rectangle has width $2 - 1 = 1$ and height 1. Its area is 1, which is also the area of the original region.

If we instead choose $v = 4 - x^2$, then $dv = -2x\,dx$ and the same integral becomes $\frac{1}{2}\int_1^4 v^{-1/2}\,dv$. The radical substitution above is especially convenient for geometry because it gives a constant height.

4.2 A curve that becomes a triangle

Example 4.3 (A radical substitution). Find the area under $y = \dfrac{\sqrt{x+1}-1}{2\sqrt{x+1}}$ on $[0, 8]$.

The expression $\sqrt{x+1}$ appears both in the numerator and in the denominator. Set

$$u = \sqrt{x+1}, \qquad du = \frac{1}{2\sqrt{x+1}}\,dx.$$

The numerator becomes $u - 1$, and the remaining factor together with dx becomes du. Since $x = 0$ gives $u = 1$ and $x = 8$ gives $u = 3$,

$$\int_0^8 \frac{\sqrt{x+1}-1}{2\sqrt{x+1}}\,dx = \int_1^3 (u-1)\,du.$$

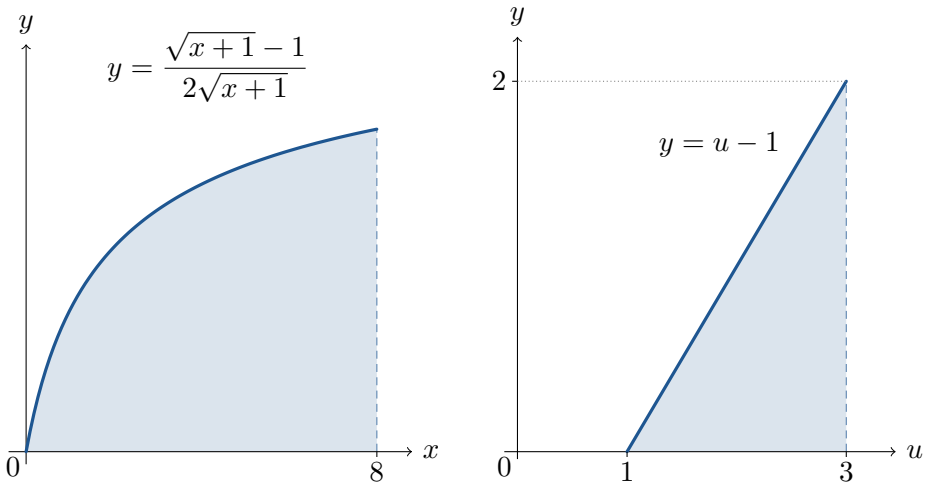


Figure 3: The region bounded by the line $y = u - 1$, the u-axis, and $u = 3$ is a right triangle. The line meets the u-axis at $u = 1$. At $u = 3$ its height is 2. The triangular region therefore has base $3 - 1 = 2$ and height 2, so

$$A = \frac{1}{2}(2)(2) = 2.$$