

Worksheet 3: Implicit Differentiation

Math 140: Calculus with Analytic Geometry

Name: _____

Date: _____

Directions. Complete the three guided questions independently and submit this worksheet in class on Monday, September 28. Show enough work to support your answers. Give exact values.

An equation involving x and y can describe several branches of a curve. To find the slope on a differentiable branch, treat y as a function $y(x)$ and differentiate both sides with respect to x . In particular,

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}, \quad \frac{d}{dx}(xy) = x\frac{dy}{dx} + y.$$

Write y' for dy/dx as you work. Before finding a slope at a point, check that the point lies on the curve.

1. A circle: the chain rule and two branches.

Consider the circle $x^2 + y^2 = 100$.

- (a) Check that $(6, 8)$ and $(6, -8)$ lie on the circle. Why does the entire circle fail to define a single function $y(x)$?
- (b) Differentiate the equation with respect to x . Explain where the factor y' appears, then solve for y' wherever $y \neq 0$.
- (c) Find the tangent line at $(6, 8)$ and the slope at $(6, -8)$. Why are the slopes different even though both points have $x = 6$?
- (d) Differentiate the equation $2x + 2yy' = 0$ again. Use the product rule on yy' , solve for y'' , and substitute your formula for y' . Use $x^2 + y^2 = 100$ to simplify. Compare the concavity of the upper and lower branches at $x = 6$.

2. An ellipse: combining the product and chain rules.

Consider $x^2 + xy + 2y^2 = 11$.

- (a) Check that $(1, 2)$ is on the ellipse. Differentiate the equation, showing the product rule step for xy , and solve for y' .
- (b) Find an equation for the tangent line at $(1, 2)$.
- (c) A horizontal tangent has slope 0. Set the numerator of y' equal to 0, substitute the resulting relation into the original equation, and find all horizontal tangent *lines*. Check that the denominator is nonzero at the points you find.
- (d) A vertical tangent can occur where the denominator of y' is 0 and its numerator is nonzero. Use the original equation to find all such points and give the equations of the vertical tangent *lines*.

3. The Folium of Descartes: an implicit curve with a loop.

The curve $x^3 + y^3 = 3xy$ has a loop in the first quadrant. A point on the loop is $(\frac{2}{3}, \frac{4}{3})$.

- (a) Verify that this point lies on the curve. Differentiate both sides with respect to x ; use the product rule on $3xy$. Solve for y' wherever the denominator is nonzero.
- (b) Find the tangent line at $(\frac{2}{3}, \frac{4}{3})$. Show how you obtained its slope.
- (c) The curve also passes through $(0, 0)$. What happens if you substitute $(0, 0)$ into your formula for y' ? Explain why this calculation alone cannot tell you a tangent line there.
- (d) To inspect the origin, take a point on the curve with $x \neq 0$ and write $y = tx$, where $t = y/x$. Substitute into $x^3 + y^3 = 3xy$ and solve for x and y in terms of t (you should obtain $x = 3t/(1 + t^3)$ and $y = 3t^2/(1 + t^3)$). As $t \rightarrow 0^+$ and as $t \rightarrow +\infty$, show that both paths approach $(0, 0)$. Use $y/x = t$ in the first limit and $x/y = 1/t$ in the second to identify the two tangent directions at the origin.