

Worksheet 5: Derivative Tests and Applications

Math 140: Calculus with Analytic Geometry

Answer Key

Directions. Complete this take-home review over several days. Use the Mean Value Theorem, first and second derivative test, and linearization notes as needed. Show sign charts and explain your conclusions; a derivative equal to zero is only a candidate. Give exact values and label important points on sketches. Use additional paper if needed. No graphing software is needed.

1. An exponential function

Let $f(x) = e^{-x^2}$, with domain $\mathbb{R}$.

- (a) Find f' and f'' , and factor each derivative. Use a sign chart for f' to find the intervals of increase and decrease and classify all critical numbers using the first derivative test.
- (b) Find the intervals of concavity and all inflection points. Check the classification of each stationary point using the second derivative test.
- (c) Find $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$, and sketch the graph, labeling intercepts, extrema, and inflection points. Does f attain an absolute maximum or minimum on $\mathbb{R}$? Explain. Identify an interval where the graph decreases while concave up, and explain what happens to its tangent slopes there.

Solution. The chain and product rules give

$$f' = -2xe^{-x^2}, \quad f'' = (4x^2 - 2)e^{-x^2} = 2(2x^2 - 1)e^{-x^2}.$$

Since the exponential is positive, f increases on $(-\infty, 0)$ and decreases on $(0, \infty)$. Its only critical number is 0, giving a local maximum at $(0, 1)$. The second derivative test agrees because $f''(0) = -2 < 0$. Put $b = 1/\sqrt{2}$. The function is concave up on $(-\infty, -b)$ and (b, ∞) and concave down on $(-b, b)$. Its inflection points are $(-b, e^{-1/2})$ and $(b, e^{-1/2})$. The only intercept is $(0, 1)$ on the vertical axis; there are no horizontal-axis intercepts. Both end limits are 0, so the graph approaches $y = 0$ from above. It is symmetric about the vertical axis, since $(-x)^2 = x^2$. The absolute maximum is 1 at 0. There is no absolute minimum: all values are positive, but they approach 0. On (b, ∞) the graph decreases while concave up; its negative tangent slopes increase toward 0.

2. A trigonometric function with a nonlinear input

Let $g(x) = \sin(x^2)$ on $[-\sqrt{\pi}, \sqrt{\pi}]$. Use radians. Only the signs and standard unit-circle values of sine and cosine are needed; no double-angle identities are needed.

- (a) Find g' and g'' . Find all interior critical numbers. Make a sign chart for g' and use the first derivative test to classify each critical number.
- (b) Evaluate g'' at each critical number and use the second derivative test to check your classifications. Explain why evaluating g'' at these points is sufficient for this test, even without finding every interval of concavity.
- (c) Compare values at the endpoints and interior critical numbers to find the absolute maximum and minimum on the given interval. Give all inputs where they occur. Sketch the graph, labeling its intercepts and extrema; a complete concavity analysis is not required.

Solution. The chain and product rules give

$$g' = 2x \cos(x^2), \quad g'' = 2 \cos(x^2) - 4x^2 \sin(x^2).$$

Put $a = \sqrt{\pi/2}$. Since $0 \leq x^2 \leq \pi$ on the domain, the interior critical numbers are $-a, 0, a$. The signs of g' on $(-\sqrt{\pi}, -a)$, $(-a, 0)$, $(0, a)$, and $(a, \sqrt{\pi})$ are $+, -, +, -$, respectively. Thus there are local maxima at $(-a, 1)$ and $(a, 1)$ and a local minimum at $(0, 0)$. At 0, $g''(0) = 2 > 0$; at $\pm a$, $g''(\pm a) = -2\pi < 0$. These values confirm the classifications. The second derivative test uses the second derivative at a stationary point; it does not require solving $g''(x) = 0$ throughout the domain. The endpoint values are $\sin \pi = 0$. The absolute maximum is 1 at $x = \pm a$; the absolute minimum is 0 at $x = -\sqrt{\pi}, 0, \sqrt{\pi}$. The horizontal-axis intercepts are $(-\sqrt{\pi}, 0)$, $(0, 0)$, and $(\sqrt{\pi}, 0)$; $(0, 0)$ is also the vertical-axis intercept. The sketch rises to the left maximum, falls to the minimum at 0, rises to the right maximum, then falls to the right endpoint. The graph is symmetric about the vertical axis because $(-x)^2 = x^2$.

3. A rational function

Let $h(x) = \frac{x^2}{x^2 + 3}$, with domain $\mathbb{R}$.

- (a) Find and simplify h' and h'' . Find all critical numbers, intervals of increase and decrease, and local extrema.
- (b) Use the second derivative to find concavity intervals and inflection points. Does an inflection point have to be a critical point of h ? Support your answer with this function.
- (c) Find $\lim_{x \rightarrow \pm\infty} h(x)$ and sketch the graph, labeling its extrema, inflection points, and horizontal asymptote. Does h attain an absolute maximum or minimum on $\mathbb{R}$? Explain.

Solution. The quotient rule gives

$$h' = \frac{6x}{(x^2 + 3)^2}, \quad h'' = \frac{18(1 - x^2)}{(x^2 + 3)^3}.$$

The function decreases on $(-\infty, 0)$ and increases on $(0, \infty)$, with minimum $(0, 0)$. It is concave down on $(-\infty, -1)$ and $(1, \infty)$ and up on $(-1, 1)$. Its inflection points are $(-1, 1/4)$ and $(1, 1/4)$; $h'(\pm 1) = \pm 3/8 \neq 0$, so these are not critical points. Since $h = 1 - 3/(x^2 + 3)$, both end limits are 1. The graph approaches the horizontal asymptote $y = 1$ from below. Its absolute minimum is 0 at 0; it has no absolute maximum because $h(x) < 1$ everywhere but values approach 1.

4. The Mean Value Theorem and a logarithm bound

- (a) State the Mean Value Theorem, including its hypotheses and conclusion.
- (b) Show that $f(x) = \ln x$ on $[1, 2]$ satisfies the hypotheses. Apply the theorem to express $\ln 2$ in terms of $f'(c)$ for some $c \in (1, 2)$.
- (c) Use $1 < c < 2$ to prove that $\frac{1}{2} < \ln 2 < 1$. Save this bound for the approximation in Problem 6.

Solution. If f is continuous on $[a, b]$ and differentiable on (a, b) , where $a < b$, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The logarithm is continuous on $[1, 2]$ and differentiable on $(1, 2)$ because $\ln x$ is differentiable for every $x > 0$. Since $f'(x) = 1/x$ and $\ln 1 = 0$, the Mean Value Theorem gives

$$\ln 2 = \frac{\ln 2 - \ln 1}{2 - 1} = f'(c) = \frac{1}{c} \quad \text{for some } 1 < c < 2.$$

Taking reciprocals of these positive numbers gives $1/2 < 1/c < 1$, so $1/2 < \ln 2 < 1$.

5. When the second derivative test is inconclusive

Consider $p(x) = (x - 1)^4$, $q(x) = (x - 1)^3$, and $r(x) = |x - 1|$.

- (a) Find the first and second derivative functions of both p and q . Evaluate them at $x = 1$. What conclusion does the second derivative test give for each function?
- (b) Use the first derivative test to classify $x = 1$ for each function. Give each extremum as a point on its graph.
- (c) Determine whether $(1, 0)$ is an inflection point of p or q . Then find r' for $x < 1$ and $x > 1$ and classify $x = 1$ for r . Explain why the second derivative test is inconclusive for p and q , but does not apply to r .

Solution. The derivatives are $p' = 4(x - 1)^3$, $p'' = 12(x - 1)^2$, $q' = 3(x - 1)^2$, and $q'' = 6(x - 1)$. All are zero at 1, so the second derivative test is inconclusive for both functions. The signs of p' are negative then positive, giving a local minimum at $(1, 0)$. The signs of q' are positive on both sides, giving no local extremum. The function p is concave up on both sides, so $(1, 0)$ is not an inflection point. The concavity of q changes from down to up there, so it is an inflection point. For r , the derivative is -1 for $x < 1$ and 1 for $x > 1$. Continuity and the first derivative test give a local minimum at $(1, 0)$, although $r'(1)$ does not exist. Thus 1 is still a critical number. The second derivative test is inconclusive for p and q because the second derivatives are zero at their stationary points; it does not apply to r because the first derivative does not exist there.

6. A logarithm and its tangent approximation

Let $k(x) = \ln(1 + x^2)$, with domain $\mathbb{R}$.

- (a) Find and simplify k' and k'' . Determine the intervals of increase and decrease and classify all critical numbers using the first derivative test.
- (b) Find the intervals of concavity and all inflection points. Check the extremum classification using the second derivative test.
- (c) Find the linearization L at $a = 1$ and use it to estimate $k(1.1) = \ln(2.21)$. Keep $\ln 2$ in your estimate. Is the estimate too high or too low? Justify using concavity between 1 and 1.1. Use the bound from Problem 4 to give lower and upper bounds for the value of your linear estimate.

Solution. The chain and quotient rules give

$$k'(x) = \frac{2x}{1+x^2}, \quad k''(x) = \frac{2(1-x^2)}{(1+x^2)^2}.$$

The function decreases on $(-\infty, 0)$ and increases on $(0, \infty)$, giving a local (and absolute) minimum at $(0, 0)$. Since $k''(0) = 2 > 0$, the second derivative test agrees. The function is concave up on $(-1, 1)$ and concave down on $(-\infty, -1)$ and $(1, \infty)$. Concavity changes at both -1 and 1 , so its inflection points are $(-1, \ln 2)$ and $(1, \ln 2)$. At $a = 1$, $k(1) = \ln 2$ and $k'(1) = 1$, so

$$L(x) = \ln 2 + (x - 1), \quad \ln(2.21) \approx L(1.1) = \ln 2 + \frac{1}{10}.$$

This estimate is too high: $k''(x) < 0$ on $(1, 1.1)$, so the graph lies below the tangent line to the right of 1. Although $k''(1) = 0$, concavity on the interval determines the comparison. Problem 4 gives

$$\frac{3}{5} < L(1.1) < \frac{11}{10}.$$

These are bounds for the linear estimate; an approximation alone does not transfer both bounds to the actual value of $k(1.1)$.

7. Position, direction, and turning

A cart moves along a straight track. Its position, measured in meters from a fixed marker, is

$$s(t) = t^3 - 6t^2 + 9t, \quad 0 \leq t \leq 4,$$

where t is measured in seconds. Positive position means the cart is to the right of the marker. Velocity is the rate of change of position; its sign tells the direction of motion. Acceleration is the rate of change of velocity.

- (a) Find the velocity $v(t) = s'(t)$ and acceleration $a(t) = s''(t)$. Make a sign chart for velocity. When does the cart move right, when does it move left, and at which interior times does it reverse direction?
- (b) Find when the position graph is concave up or down and locate its inflection point. On $(1, 2)$ and $(2, 3)$ the cart moves left. On which interval does its velocity become more negative, and on which does it become less negative? Explain using acceleration.
- (c) Compare the positions at the endpoints and interior critical times to find the leftmost and rightmost positions during $[0, 4]$. Give every time at which each is attained. Explain why being momentarily at rest is different from being at the marker.

Solution. The derivatives are $v = 3(t - 1)(t - 3)$ m/s and $a = 6(t - 2)$ m/s². Velocity is positive on $(0, 1)$ and $(3, 4)$ and negative on $(1, 3)$, so the cart moves right, left, then right. It reverses at 1 s and 3 s. The position graph is concave down on $(0, 2)$ and up on $(2, 4)$, with inflection point $(2, 2)$. On $(1, 2)$ acceleration is negative and velocity becomes more negative; on $(2, 3)$ acceleration is positive and velocity becomes less negative. At 2 s the cart is still moving left, since $v(2) = -3$ m/s. The comparison is $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, and $s(4) = 4$. The leftmost position is 0 m at 0 s and 3 s; the rightmost is 4 m at 1 s and 4 s. At rest means $v = 0$; at the marker means $s = 0$. For example, the cart is at rest at 1 s but its position is 4 m.

8. Building a model for maximum area

A farmer has 80 m of fencing for a rectangular pen beside a straight river. No fence is needed along the river; all 80 m is used for the other three sides. Let x be the length of each of the two sides perpendicular to the river, and let y be the length of the fenced side parallel to it. Both lengths must be positive. The goal is to choose the dimensions that enclose the greatest area. This is an optimization problem: first express the quantity to be maximized as a function of one variable, then use derivatives to compare possible choices.

- (a) Draw and label the pen, marking the unfenced side. Write the equation for the total fencing length. Solve it for y and substitute into $A = xy$ to express area as $A(x)$. Find the allowed interval for x using $x > 0$ and $y > 0$.
- (b) Find A' and A'' . Find the critical number in the allowed interval. Use derivative signs on each side to show that this choice gives the greatest area, and check with the second derivative test. Give both dimensions and the area, including units.
- (c) For an independent check, extend $A(x)$ continuously to the endpoints of the corresponding closed interval. Compare its values at both endpoints and the critical number. What shape do the endpoints represent, and why does the maximizing choice still belong to the original physical domain?

Solution. The fencing constraint is $2x + y = 80$, so $y = 80 - 2x$ and $A(x) = 80x - 2x^2$, with $0 < x < 40$. The derivatives are $A' = 80 - 4x$ and $A'' = -4$. The only critical number is 20. Area increases for $0 < x < 20$ and decreases for $20 < x < 40$, so this is an absolute maximum on the physical domain; the second derivative test also gives a local maximum. The dimensions are 20 m perpendicular to the river and 40 m parallel to it, giving 800 m^2 . The continuous extension to $[0, 40]$ has $A(0) = A(40) = 0$ and $A(20) = 800$. This comparison again proves an absolute maximum. At either endpoint one dimension vanishes, producing a degenerate rectangle. At 20 both dimensions are positive, so the maximizing pen is feasible.