Worksheet 6: Curve Sketching and Optimization

Math 140: Calculus with Analytic Geometry

Answer Key

Directions. Show your work, justify your conclusions, and give exact answers with units. Label sketches clearly. For optimization problems, state the objective, constraints, and feasible domain, and justify an absolute optimum. No external resources are allowed.

1. **A rational curve with a hole and an asymptote**

   Let $f(x) = \dfrac{x^2 + x - 2}{x^2 - 1}$, with its original domain.

   (a) State the domain and find all intercepts. Simplify the formula without changing the domain. At each excluded input, evaluate the one-sided limits and decide whether the graph has a hole or a vertical asymptote. Give the coordinates of any hole and identify all horizontal asymptotes.

   (b) Find $f'(x)$ and $f''(x)$ on the domain. Determine intervals of increase, decrease, and concavity, splitting intervals at every excluded input. Locate any critical numbers, local extrema, and inflection points. Explain why a change in concavity across a missing input does not give an inflection point.

   (c) Sketch every branch, labeling intercepts, holes, and asymptotes. Does the graph attain the height of its horizontal asymptote? Justify your answer algebraically.

   Solution. The original domain is $\mathbb{R} \setminus \{-1, 1\}$. Factoring gives

   $$f(x) = \frac{(x+2)(x-1)}{(x-1)(x+1)} = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}, \qquad x \neq -1, 1.$$

   The intercepts are $(-2, 0)$ and $(0, 2)$. At -1, the left limit is $-\infty$ and the right limit is $+\infty$, giving the vertical asymptote $x = -1$. Both one-sided limits at 1 equal $3/2$, so the hole is $(1, 3/2)$. Both end limits are 1, giving the horizontal asymptote $y = 1$.

   $$f'(x) = -\frac{1}{(x+1)^2}, \qquad f''(x) = \frac{2}{(x+1)^3}, \qquad x \neq -1, 1.$$

   The function decreases on each of $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$; it increases on none. There are no critical numbers or local extrema: f' never vanishes or fails to exist at an input in the domain. The graph is concave down on $(-\infty, -1)$ and concave up on $(-1, 1)$ and $(1, \infty)$. There are no inflection points, since the concavity change occurs across an excluded input. The left branch decreases from the horizontal asymptote through $(-2, 0)$ to $-\infty$. The right branch decreases from $+\infty$ through $(0, 2)$ toward $y = 1$, with an open circle at $(1, 3/2)$. Finally, $f(x) = 1$ would require $1/(x+1) = 0$, which is impossible.

2. **A logarithmic curve**

Let $g(x) = (\ln x)^2$.

(a) State the domain and find all intercepts. Evaluate the limits at the ends of the domain and identify any vertical or horizontal asymptotes.

(b) Find and factor $g\prime$ and $g\prime\prime$. Determine intervals of increase, decrease, and concavity. Locate all local extrema and inflection points.

(c) Sketch the graph and decide whether it has an absolute maximum or minimum. Explain how the graph can increase while concave down, and identify an interval on which this occurs.

Solution.

The domain is $(0, \infty)$ and the only intercept is $(1, 0)$; there is no y-intercept. Both $\lim_{x\to 0^+} g(x)$ and $\lim_{x\to\infty} g(x)$ equal $+\infty$. Thus $x = 0$ is a vertical asymptote and there is no horizontal asymptote.

$$g'(x) = \frac{2\ln x}{x}, \qquad g''(x) = \frac{2(1 - \ln x)}{x^2}.$$

The function decreases on $(0, 1)$ and increases on $(1, \infty)$. Its local and absolute minimum is $(1, 0)$, since $g(x) \geq 0$, and it has no absolute maximum. It is concave up on $(0, e)$ and concave down on (e, ∞), with inflection point $(e, 1)$. The sketch falls from $+\infty$ at 0^+ to $(1, 0)$, then rises without bound, changing concavity at $(e, 1)$. On (e, ∞) the function increases while concave down: its slopes are positive but decreasing.

3. **A critical number where the derivative does not exist**

Let $h(x) = x^{2/3}(x-5)$ on $\mathbb{R}$, where $x^{2/3} = (\sqrt[3]{x})^2$.

(a) Find $h\prime(x)$ for $x \neq 0$ and use the difference quotient at 0 to show that $h\prime(0)$ does not exist. Find all critical numbers and use derivative signs to classify them.

(b) Find $h\prime\prime(x)$ for $x \neq 0$. Determine concavity intervals and all inflection points. Explain why a zero or undefined second derivative is only a candidate for an inflection point.

(c) Find the intercepts and both limits at infinity, then sketch the graph. Label all local extrema and inflection points. Does the function attain an absolute maximum or minimum on $\mathbb{R}$?

Solution.

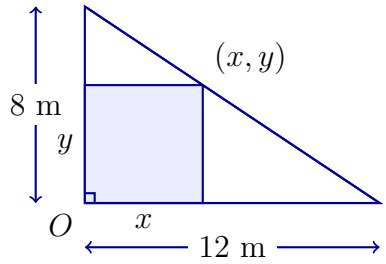
For $x \neq 0$,

$$h'(x) = \frac{5(x-2)}{3\sqrt[3]{x}}, \qquad h''(x) = \frac{10(x+1)}{9x^{4/3}}.$$

At 0, the difference quotient is $(t-5)/\sqrt[3]{t}$, which tends to $+\infty$ from the left and $-\infty$ from the right. Hence no finite derivative exists there. The critical numbers are 0 and 2. The signs of h' on $(-\infty, 0), (0, 2), (2, \infty)$ are $+, -, +$. Continuity at 0 and these signs give a local maximum $(0, 0)$ and local minimum $(2, -3\sqrt[3]{4})$. The graph is concave down on $(-\infty, -1)$ and concave up on $(-1, 0)$ and $(0, \infty)$. The only inflection point is $(-1, -6)$. Concavity does not change at 0, even though h'' is undefined there. The intercepts are $(0, 0)$ and $(5, 0)$. The end limits are $-\infty$ on the left and $+\infty$ on the right, so neither absolute extremum exists. The sketch rises from the lower left, changes concavity at $(-1, -6)$, reaches a sharp peak at $(0, 0)$, falls to its local minimum at 2, then rises through $(5, 0)$ to $+\infty$.

4. **A rectangle inside a right triangle**

A right triangle has vertices $(0,0)$, $(12,0)$, and $(0,8)$, with coordinates in meters. A rectangle has one vertex at the origin, two sides on the coordinate axes, and its opposite vertex (x,y) on the hypotenuse.



(a) Use similar triangles or the equation of the hypotenuse to write a model for maximizing the rectangle's area. Include positivity constraints and explain how the geometry relates x and y.

(b) Reduce the area to one variable and state its feasible domain. Find all feasible critical numbers.

(c) Justify an absolute maximum and give the dimensions and maximum area. Explain what the boundary configurations represent.

Solution.

The hypotenuse gives $y = 8 - 2x/3$. The model is

$$\begin{aligned} \max_{x,\,y} \quad & A = xy \\ \text{s.t.} \quad & 2x + 3y = 24, \\ & x > 0, \quad y > 0 \end{aligned}$$

Thus $A(x) = 8x - 2x^2/3$ on $(0,12)$, with $A'(x) = 8 - 4x/3$. The only critical number is 6; the derivative is positive before 6 and negative after it throughout the domain. The maximum dimensions are 6 m by 4 m, giving 24 m^2. The continuous extension has $A(0) = A(12) = 0$; these collapsed rectangles cannot improve on the feasible interior optimum.

5. **Minimum construction cost for an open tank**

An open-top tank with a square base must hold 54 ft^3. The base material costs \$1 per square foot and the material for each of the four vertical sides costs \$2 per square foot. Ignore seams, thickness, and waste.

(a) Define the base side length and height. Write a model that minimizes total material cost subject to the volume and domain constraints. Account for every face that is present.

(b) Express cost as a function of the base side length alone. State the domain and find all feasible critical numbers.

(c) Justify an absolute minimum on the entire domain. Give the base dimensions, height, and minimum cost. Explain why checking only the second derivative at the critical number would not by itself prove the global conclusion.

Solution.

Let $s, h > 0$ be measured in feet. The base costs s^2 dollars and the four sides cost $2(4sh) = 8sh$ dollars.

$$\begin{aligned} \min_{s,\,h} \quad & C = s^2 + 8sh \\ \text{s.t.} \quad & s^2 h = 54, \\ & s > 0, \quad h > 0 \end{aligned}$$

Then $C(s) = s^2 + 432/s$ for $s > 0$, and

$$C'(s) = \frac{2(s^3 - 216)}{s^2}.$$

It is negative on $(0, 6)$ and positive on $(6, \infty)$, so $s = 6$ gives the absolute minimum. Then $h = 54/36 = 3/2$. The base is 6 ft by 6 ft, the height is 3/2 ft, and the minimum cost is \$108. Both boundary limits of C are $+\infty$. The second derivative test at one point only establishes a local minimum; the derivative signs across the entire domain establish the absolute minimum.

6. **The shortest distance to a line**

Find the point on the line $2x + y = 6$ that is closest to the origin. Coordinates are measured in meters, and the entire line is available.

(a) Write a distance-minimization model with the line equation as a constraint. Explain why minimizing squared distance gives exactly the same minimizing points.

(b) Use the constraint to express squared distance as a function of x on its full domain. Find the critical number.

(c) Prove that your point gives the absolute minimum distance. Give the point and the distance, then check geometrically that the segment from the origin is perpendicular to the line.

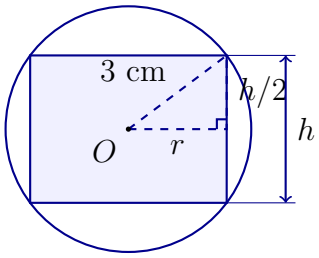
Solution.

$$\begin{aligned} \min_{x,\,y} \quad & D = \sqrt{x^2 + y^2} \\ \text{s.t.} \quad & 2x + y = 6, \\ & x, y \in \mathbb{R} \end{aligned}$$

The square root is strictly increasing on $[0, \infty)$, so minimizing D^2 is equivalent. Substitute $y = 6 - 2x$ to get $F(x) = 5x^2 - 24x + 36$ on $\mathbb{R}$. Its derivative is $F'(x) = 10x - 24$, negative before $12/5$ and positive after it. Thus the unique closest point is $(12/5, 6/5)$ and the distance is $6/\sqrt{5} = 6\sqrt{5}/5$ m. Alternatively, $F(x) = 5(x - 12/5)^2 + 36/5$ proves the absolute minimum directly. The line has slope -2, and the segment to $(12/5, 6/5)$ has slope $1/2$, so they are perpendicular.

7. **A cylinder inside a sphere**

A right circular cylinder is inscribed in a sphere of radius 3 cm. The cylinder and sphere have the same center, and both circular rims of the cylinder lie on the sphere. Find the cylinder of maximum volume.



(a) Use the axial cross-section shown. Let r and h be the cylinder's radius and full height. Use the Pythagorean Theorem to write a model for maximizing volume, including the geometric and domain constraints.

(b) Eliminate r to express volume as a function of h. State the domain and find all feasible critical numbers.

(c) Justify an absolute maximum. Give the radius, full height, and maximum volume. Explain why the cross-section uses $h/2$, rather than h.

Solution.

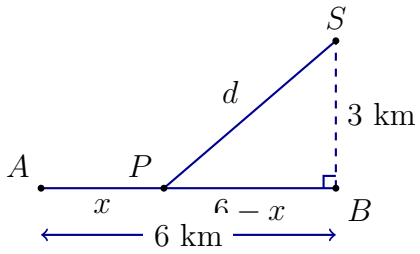
The right triangle has legs r and $h/2$ and hypotenuse 3.

$$\begin{aligned} \max_{r,\,h} \quad & V = \pi r^2 h \\ \text{s.t.} \quad & r^2 + h^2/4 = 9, \\ & 0 < r < 3, \quad 0 < h < 6 \end{aligned}$$

Thus $V(h) = \pi(9h - h^3/4)$ on $(0, 6)$ and $V'(h) = \pi(9 - 3h^2/4)$. The unique feasible critical number is $h = 2\sqrt{3}$. The derivative is positive before it and negative afterward, proving an absolute maximum; the continuous extension also has zero volume at 0 and 6. Then $r = \sqrt{6}$ cm, $h = 2\sqrt{3}$ cm, and $V = 12\sqrt{3}\pi$ cm^3. The center bisects the full height, so the triangle from the sphere's center to a rim uses the half-height $h/2$.

8. **Choosing where to enter the water**

A lifeguard starts at point A on a straight shoreline. Point B is 6 km along the shore from A, and a swimmer is at point S, 3 km directly offshore from B. The lifeguard runs from A toward B to a point P on the segment AB, then swims in a straight line from P to S. Assume constant speeds of 4 km/h on land and 2 km/h in water.



(a) Let $x = AP$ and let $d = PS$. Write a model that minimizes total travel time, including the distance relation and domain constraints. Explain why the distances must be divided by their respective speeds.

(b) Express travel time as a function of x alone. Differentiate and find all critical numbers in the interior of the feasible interval. Check any candidates produced by squaring against the original derivative equation.

(c) Justify an absolute minimum, accounting for the endpoints. Give the optimal running and swimming distances and the minimum time. Compare with running all the way to B before swimming.

Solution.

Since $PB = 6 - x$, the model is

$$\begin{aligned} \min_{x,\,d} \quad & T = x/4 + d/2 \\ \text{s.t.} \quad & d^2 = (6-x)^2 + 9, \\ & 0 \le x \le 6, \quad d > 0 \end{aligned}$$

Time is distance divided by speed, so T is in hours. Elimination gives

$$T(x) = \frac{x}{4} + \frac{\sqrt{(6-x)^2+9}}{2}, \quad 0 \le x \le 6, \qquad T'(x) = \frac{1}{4} - \frac{6-x}{2\sqrt{(6-x)^2+9}}.$$

Writing $u = 6 - x \ge 0$, the stationary equation is $\sqrt{u^2+9} = 2u$. Squaring gives $u^2 = 3$; only $u = \sqrt{3}$ satisfies the original equation. Thus $x = 6 - \sqrt{3}$. Also,

$$T''(x) = \frac{9}{2((6-x)^2+9)^{3/2}} > 0$$

throughout $[0, 6]$, so T' is strictly increasing, negative before the critical number and positive after it. This proves an absolute minimum including the endpoints. The optimal running distance is $6 - \sqrt{3}$ km, the swimming distance is $2\sqrt{3}$ km, and the minimum time is

$$T_{\min} = \frac{3}{2} + \frac{3\sqrt{3}}{4} \text{ hours.}$$

The endpoint times are $T(0) = 3\sqrt{5}/2$ and $T(6) = 3$ hours. Running to B and then swimming takes 3 hours, longer than the optimum since $\sqrt{3} < 2$.