Math 140 Worksheet 10 Solutions

Week 10: Curve Sketching

1. Let

$$f(x) = \frac{x^2}{x-1}.$$

We analyze the function using the curve sketching checklist.

Domain.

$$\text{Domain} = (-\infty, 1) \cup (1, \infty),$$

since the denominator cannot be zero.

Intercepts. To find the x-intercepts, solve

$$\frac{x^2}{x-1} = 0.$$

This occurs when $x^2 = 0$, so $x = 0$. Thus the only x-intercept is $(0, 0)$.

The y-intercept is

$$f(0) = \frac{0^2}{0-1} = 0,$$

so the y-intercept is also $(0, 0)$.

Vertical asymptote. Since the denominator is zero at $x = 1$, there is a vertical asymptote at

$$x = 1.$$

Also,

$$\lim_{x\to 1^-} \frac{x^2}{x-1} = -\infty, \qquad \lim_{x\to 1^+} \frac{x^2}{x-1} = +\infty.$$

End behavior / slant asymptote. Use polynomial division:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Thus,

$$f(x) = x + 1 + \frac{1}{x-1}.$$

As $x \to \pm\infty$, the term $\frac{1}{x-1} \to 0$, so the graph approaches the slant asymptote

$$y = x + 1.$$

First derivative.

$$f(x) = x + 1 + \frac{1}{x-1}$$

gives

$$f'(x) = 1 - \frac{1}{(x-1)^2} = \frac{(x-1)^2 - 1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Critical numbers. Critical numbers occur when $f'(x) = 0$ or undefined, provided f is defined there.

From

$$\frac{x(x-2)}{(x-1)^2} = 0,$$

we get

$$x = 0, \qquad x = 2.$$

The derivative is undefined at $x = 1$, but f is not defined there, so $x = 1$ is not a critical number.

Increasing/decreasing intervals. Since $(x-1)^2 > 0$ for all $x \neq 1$, the sign of $f'(x)$ is determined by $x(x-2)$.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$x(x-2)$	$+$	$-$	$-$	$+$

Therefore, f is increasing on

$$(-\infty, 0) \cup (2, \infty),$$

and decreasing on

$$(0, 1) \cup (1, 2).$$

Local maxima and minima. At $x = 0$, f' changes from positive to negative, so there is a local maximum at

$$f(0) = 0.$$

Thus, the local maximum is $(0, 0)$.

At $x = 2$, f' changes from negative to positive, so there is a local minimum at

$$f(2) = \frac{4}{1} = 4.$$

Thus, the local minimum is $(2, 4)$.

Second derivative.

$$f'(x) = 1 - (x-1)^{-2}$$

so

$$f''(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}.$$

Concavity and inflection points. If $x < 1$, then $(x-1)^3 < 0$, so

$$f''(x) < 0.$$

Thus, the graph is concave down on

$$(-\infty, 1).$$

If $x > 1$, then $(x-1)^3 > 0$, so

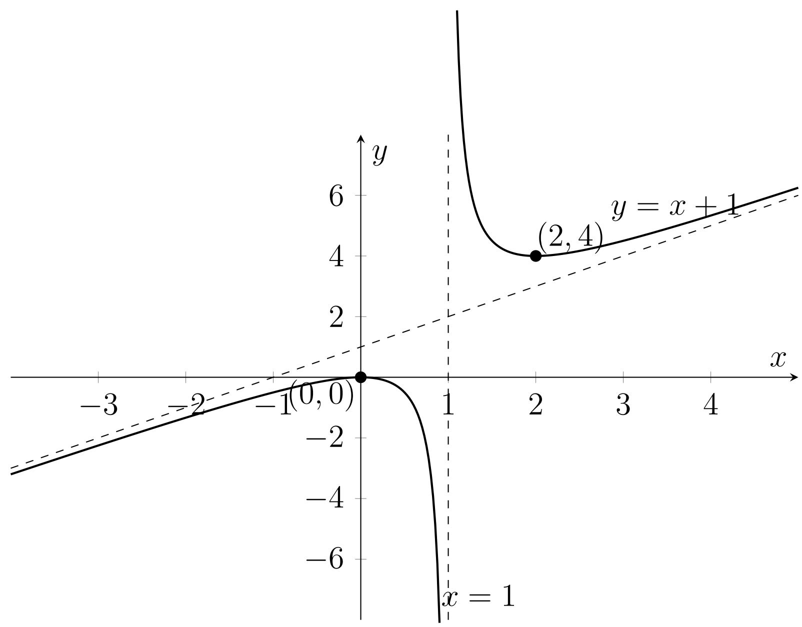
$$f''(x) > 0.$$

Thus, the graph is concave up on

$$(1, \infty).$$

There is no inflection point, since $x = 1$ is not in the domain.

Graph.



Summary. The graph has domain $(-\infty, 1) \cup (1, \infty)$, intercept $(0, 0)$, vertical asymptote $x = 1$, slant asymptote $y = x + 1$, local maximum $(0, 0)$, and local minimum $(2, 4)$. It is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

2. Let

$$f(x) = \frac{\ln x}{x}.$$

We again analyze the function using the curve sketching checklist.

Domain.

$$\text{Domain} = (0, \infty),$$

since $\ln x$ is only defined for $x > 0$.

Intercepts. To find the x-intercept, solve

$$\frac{\ln x}{x} = 0.$$

Since $x > 0$, this occurs when

$$\ln x = 0,$$

so

$$x = 1.$$

Thus, the x-intercept is $(1, 0)$.

There is no y-intercept because the function is not defined at $x = 0$.

Vertical asymptote. As $x \to 0^+$,

$$\ln x \to -\infty \quad \text{and} \quad x \to 0^+,$$

so

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty.$$

Thus, there is a vertical asymptote at

$$x = 0.$$

Horizontal asymptote / end behavior. As $x \to \infty$,

$$\frac{\ln x}{x} \to 0.$$

Therefore, the graph has horizontal asymptote

$$y = 0.$$

First derivative. Use the quotient rule:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

Critical numbers. Critical numbers occur when $f'(x) = 0$ or undefined, provided f is defined there.

Since $x^2 > 0$ for $x > 0$, we set

$$1 - \ln x = 0.$$

Thus,

$$\ln x = 1 \implies x = e.$$

So the only critical number is

$$x = e.$$

Increasing/decreasing intervals. Because $x^2 > 0$ for $x > 0$, the sign of $f'(x)$ is determined by $1 - \ln x$.

If $0 < x < e$, then $\ln x < 1$, so $f'(x) > 0$. If $x > e$, then $\ln x > 1$, so $f'(x) < 0$.

Therefore, f is increasing on

$$(0, e)$$

and decreasing on

$$(e, \infty).$$

Local maximum and minimum. At $x = e$, the derivative changes from positive to negative, so f has a local maximum at

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}.$$

Thus, the local maximum is

$$\left(e, \frac{1}{e}\right).$$

There is no local minimum.

Second derivative. Starting from

$$f'(x) = \frac{1 - \ln x}{x^2} = (1 - \ln x)x^{-2},$$

we differentiate:

$$f''(x) = \left(-\frac{1}{x}\right)x^{-2} + (1 - \ln x)(-2)x^{-3}.$$

So

$$f''(x) = \frac{-1 - 2(1 - \ln x)}{x^3} = \frac{2\ln x - 3}{x^3}.$$

Concavity and inflection points. Since $x^3 > 0$ for $x > 0$, the sign of $f''(x)$ is determined by $2\ln x - 3$.

Set

$$2\ln x - 3 = 0 \implies \ln x = \frac{3}{2} \implies x = e^{3/2}.$$

If $0 < x < e^{3/2}$, then $2\ln x - 3 < 0$, so the graph is concave down.

If $x > e^{3/2}$, then $2\ln x - 3 > 0$, so the graph is concave up.

Thus, the graph is concave down on

$$(0, e^{3/2})$$

and concave up on

$$(e^{3/2}, \infty).$$

Since the concavity changes, there is an inflection point at

$$x = e^{3/2}.$$

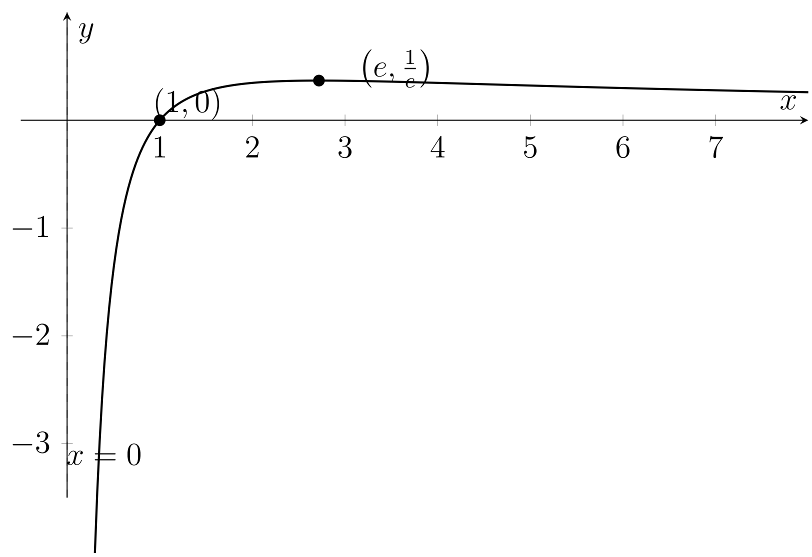
Its y-coordinate is

$$f\left(e^{3/2}\right) = \frac{\ln\left(e^{3/2}\right)}{e^{3/2}} = \frac{3/2}{e^{3/2}}.$$

So the inflection point is

$$\left(e^{3/2}, \frac{3}{2e^{3/2}}\right).$$

Graph.



Summary. The graph has domain $(0, \infty)$, x-intercept $(1, 0)$, vertical asymptote $x = 0$, horizontal asymptote $y = 0$, and local maximum $\left(e, \frac{1}{e}\right)$. It is increasing on $(0, e)$, decreasing on (e, ∞), concave down on $(0, e^{3/2})$, and concave up on $(e^{3/2}, \infty)$.