Math 140 Worksheet 11 Solutions

Week 11: Optimization

1. Find the dimensions of the isoceles triangle of maximum area with perimeter 30.

   Let the base be b and let the two congruent sides be s. Then

$$b + 2s = 30.$$

   We want to maximize the area.

   Let h be the height of the triangle. Then the height bisects the base, so by the Pythagorean Theorem,

$$h = \sqrt{s^2 - \left(\frac{b}{2}\right)^2}.$$

   Thus the area is

$$A = \frac{1}{2}bh = \frac{1}{2}b\sqrt{s^2 - \left(\frac{b}{2}\right)^2}.$$

   Since $b + 2s = 30$, we have

$$s = \frac{30 - b}{2}.$$

   Substitute into the area formula:

$$A(b) = \frac{1}{2}b\sqrt{\left(\frac{30 - b}{2}\right)^2 - \left(\frac{b}{2}\right)^2}.$$

   Simplify:

$$A(b) = \frac{1}{2}b\sqrt{\frac{(30 - b)^2 - b^2}{4}} = \frac{b}{4}\sqrt{900 - 60b}.$$

   To simplify the optimization, maximize $A(b)^2$ instead:

$$A(b)^2 = \frac{b^2}{16}(900 - 60b).$$

   So it is enough to maximize

$$f(b) = b^2(900 - 60b) = 900b^2 - 60b^3.$$

   Differentiate:

$$f'(b) = 1800b - 180b^2 = 180b(10 - b).$$

   Critical numbers are

$$b = 0, \qquad b = 10.$$

   Since $b = 0$ gives zero area, the maximum occurs at

$$b = 10.$$

Then

$$s = \frac{30 - 10}{2} = 10.$$

Therefore, the isosceles triangle of maximum area is actually an equilateral triangle.

$$\boxed{\text{Base } = 10, \quad \text{equal sides } = 10}$$

2. Find the rectangle of maximum area that can be inscribed in a equilateral triangle with side length 15.

   Let the equilateral triangle have side length 15. Its height is

$$H = \frac{\sqrt{3}}{2}(15) = \frac{15\sqrt{3}}{2}.$$

   Let the inscribed rectangle have width w and height h. Since the rectangle sits inside the triangle, its top corners lie on the sloping sides.

   By similar triangles, the horizontal width of the triangle at height h above the base is

$$w = 15\left(1 - \frac{h}{H}\right).$$

   Thus the area of the rectangle is

$$A(h) = wh = 15h\left(1 - \frac{h}{H}\right) = 15h - \frac{15}{H}h^2.$$

   Differentiate:

$$A'(h) = 15 - \frac{30}{H}h.$$

   Set $A'(h) = 0$:

$$15 - \frac{30}{H}h = 0 \quad \Longrightarrow \quad h = \frac{H}{2}.$$

   Hence

$$h = \frac{1}{2} \cdot \frac{15\sqrt{3}}{2} = \frac{15\sqrt{3}}{4}.$$

   Then

$$w = 15\left(1 - \frac{1}{2}\right) = \frac{15}{2}.$$

   Therefore, the rectangle of maximum area has dimensions

$$\boxed{\frac{15}{2} \text{ by } \frac{15\sqrt{3}}{4}}.$$

3. A right circular cone has height 10 and radius 6. Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in the cone.

   Let the cylinder have radius r and height h.

   By similar triangles, the radius of the cone at height h above the base is

$$r = 6\left(1 - \frac{h}{10}\right).$$

   So the cylinder volume is

$$V = \pi r^2 h = \pi\left(6\left(1 - \frac{h}{10}\right)\right)^2 h.$$

   Simplify:

$$V(h) = 36\pi h\left(1 - \frac{h}{10}\right)^2.$$

   Expand:

$$V(h) = 36\pi h\left(1 - \frac{h}{5} + \frac{h^2}{100}\right) = 36\pi\left(h - \frac{h^2}{5} + \frac{h^3}{100}\right).$$

   Differentiate:

$$V'(h) = 36\pi\left(1 - \frac{2h}{5} + \frac{3h^2}{100}\right).$$

   Set $V'(h) = 0$:

$$1 - \frac{2h}{5} + \frac{3h^2}{100} = 0.$$

   Multiply by 100:

$$100 - 40h + 3h^2 = 0.$$

   So

$$3h^2 - 40h + 100 = 0.$$

   Using the quadratic formula,

$$h = \frac{40 \pm \sqrt{1600 - 1200}}{6} = \frac{40 \pm 20}{6}.$$

   Thus

$$h = 10 \quad \text{or} \quad h = \frac{10}{3}.$$

   The value $h = 10$ gives $r = 0$, so the maximum occurs at

$$h = \frac{10}{3}.$$

   Then

$$r = 6\left(1 - \frac{1}{3}\right) = 6 \cdot \frac{2}{3} = 4.$$

Therefore, the cylinder of maximum volume has dimensions

$$\boxed{r = 4, \quad h = \frac{10}{3}}.$$

4. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$.

   A point on the curve has the form

$$(x, \sqrt{x}), \qquad x \geq 0.$$

   The distance to $(4, 0)$ is

$$d = \sqrt{(x-4)^2 + (\sqrt{x})^2} = \sqrt{(x-4)^2 + x}.$$

   To minimize distance, it is enough to minimize

$$D(x) = (x-4)^2 + x.$$

   Expand:

$$D(x) = x^2 - 8x + 16 + x = x^2 - 7x + 16.$$

   Differentiate:

$$D'(x) = 2x - 7.$$

   Set equal to zero:

$$2x - 7 = 0 \quad \Longrightarrow \quad x = \frac{7}{2}.$$

   Then

$$y = \sqrt{\frac{7}{2}}.$$

   Therefore, the point on the curve closest to $(4, 0)$ is

$$\boxed{\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)}.$$

5. An offshore oil well is located 2 km from the closest point on a straight shoreline. A refinery is located 10 km along the shore from that point. The pipeline costs $1000000/km$ underwater and $500000/km$ on land. Where should the underwater pipeline intersect the shoreline to minimize total cost?

   Let x be the distance along the shoreline from the closest point to the point where the underwater pipeline reaches the shore. Then the underwater distance is

$$\sqrt{x^2 + 2^2} = \sqrt{x^2 + 4},$$

and the land distance is

$$10 - x.$$

Thus the total cost is

$$C(x) = 1000000\sqrt{x^2+4} + 500000(10 - x), \qquad 0 \le x \le 10.$$

Differentiate:

$$C'(x) = 1000000 \cdot \frac{x}{\sqrt{x^2+4}} - 500000.$$

Set $C'(x) = 0$:

$$1000000 \cdot \frac{x}{\sqrt{x^2+4}} = 500000.$$

Divide by 500000:

$$2\frac{x}{\sqrt{x^2+4}} = 1.$$

So

$$2x = \sqrt{x^2+4}.$$

Square both sides:

$$4x^2 = x^2 + 4.$$

Hence

$$3x^2 = 4 \quad \Longrightarrow \quad x = \frac{2}{\sqrt{3}}.$$

Since $x > 0$, we take

$$x = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Therefore, the underwater pipeline should intersect the shoreline

$$\boxed{\frac{2}{\sqrt{3}}\text{ km} = \frac{2\sqrt{3}}{3}\text{ km}}$$

from the point on the shore closest to the oil well, in the direction of the refinery.