Math 140 Worksheet 12 Solutions

We approximate the area under the curve $f(x) = x^2$ on the interval $[0, 2]$. If we divide the interval into n subintervals of equal width, then

$$\Delta x = \frac{2}{n}.$$

The partition points are

$$x_i = \frac{2i}{n}, \qquad i = 0, 1, \ldots, n.$$

Right endpoint approximation.
When $n = 4$, we have $\Delta x = \frac{1}{2}$. The right endpoints are

$$\frac{1}{2},\ 1,\ \frac{3}{2},\ 2.$$

Thus,

$$A_4^{\text{right}} = \sum_{i=1}^{4} f(x_i)\Delta x = \left[\left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2 + 2^2\right]\frac{1}{2} = \frac{15}{4}.$$

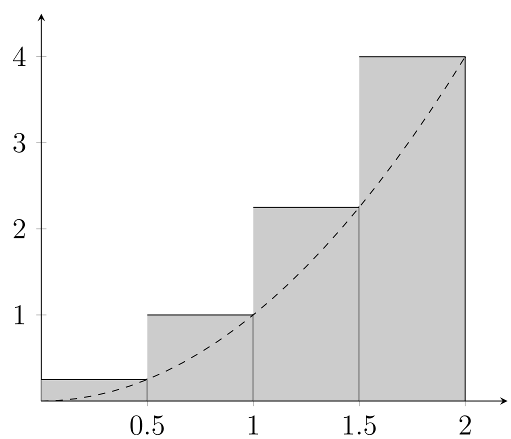

Figure 1: Right endpoint approximation with $n = 4$.

Since $f(x) = x^2$ is increasing on $[0, 2]$, the right endpoint approximation gives an overestimate of the true area.

For general n, using the right endpoints $x_i = \frac{2i}{n}$, we obtain

$$A_n^{\text{right}} = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2 \frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Left endpoint approximation.

When $n = 4$, the left endpoints are

$$0,\ \frac{1}{2},\ 1,\ \frac{3}{2}.$$

Thus,

$$A_4^{\text{left}} = \left[0^2 + \left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2\right]\frac{1}{2} = \frac{7}{4}.$$

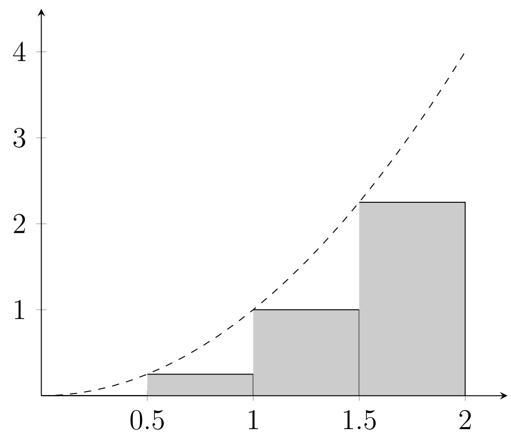

Figure 2: Left endpoint approximation with $n = 4$.

Since $f(x) = x^2$ is increasing on $[0, 2]$, the left endpoint approximation gives an underestimate of the true area.

For general n, using the left endpoints $x_{i-1} = \frac{2(i-1)}{n}$, we have

$$A_n^{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2\frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2.$$

Midpoint approximation.

When $n = 4$, the midpoints are

$$\frac{1}{4},\ \frac{3}{4},\ \frac{5}{4},\ \frac{7}{4}.$$

Thus,

$$A_4^{\text{mid}} = \left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{5}{4}\right)^2 + \left(\frac{7}{4}\right)^2\right]\frac{1}{2} = \frac{21}{8}.$$

For general n, the midpoint of the i-th subinterval is

$$m_i = \frac{2i-1}{n}.$$

Thus,

$$A_n^{\text{mid}} = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2i-1}{n}\right)^2\frac{2}{n} = \frac{2}{n^3}\sum_{i=1}^{n}(2i-1)^2.$$

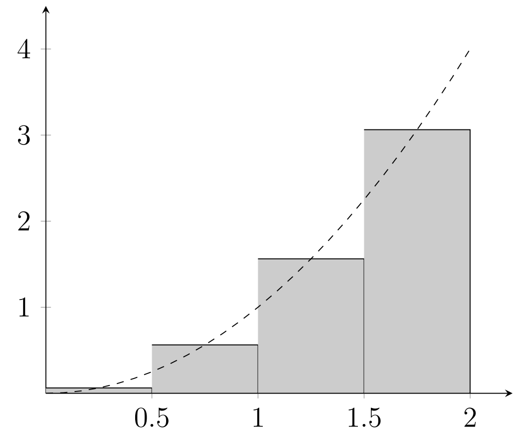

Figure 3: Midpoint approximation with $n = 4$.

Comparison.

The exact value of the area is

$$\int_0^2 x^2\,dx = \frac{8}{3}.$$

For $n = 4$, we have

$$A_4^{\text{left}} = \frac{7}{4} \;<\; \frac{8}{3} \;<\; A_4^{\text{right}} = \frac{15}{4},$$

and

$$A_4^{\text{mid}} = \frac{21}{8}$$

is closer to the exact value.