Math 140 Worksheet 8 — Solution Key

1. Let $f(x) = x^3 - 6x^2 + 9x$ on $[0, 3]$.

   (a) Since f is a polynomial, it is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Also,
   $$f(0) = 0^3 - 6(0)^2 + 9(0) = 0$$
   and
   $$f(3) = 3^3 - 6(3)^2 + 9(3) = 27 - 54 + 27 = 0.$$
   Thus $f(0) = f(3)$, so f satisfies all hypotheses of Rolle's Theorem on $[0, 3]$.

   (b) By Rolle's Theorem, there exists at least one $c \in (0, 3)$ such that $f'(c) = 0$. Compute
   $$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3).$$
   So
   $$f'(x) = 0 \quad \Longrightarrow \quad x = 1 \text{ or } x = 3.$$
   Since c must lie in $(0, 3)$, the value guaranteed by Rolle's Theorem is
   $$\boxed{c = 1}.$$

2. Let $f(x) = \ln(x)$ on $[1, e^2]$.

   (a) The function $\ln(x)$ is continuous for $x > 0$, so it is continuous on $[1, e^2]$ and differentiable on $(1, e^2)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem on $[1, e^2]$.

   (b) By the Mean Value Theorem, there exists $c \in (1, e^2)$ such that
   $$f'(c) = \frac{f(e^2) - f(1)}{e^2 - 1}.$$
   Now
   $$f(e^2) = \ln(e^2) = 2 \qquad \text{and} \qquad f(1) = \ln(1) = 0,$$
   so
   $$\frac{f(e^2) - f(1)}{e^2 - 1} = \frac{2}{e^2 - 1}.$$
   Since
   $$f'(x) = \frac{1}{x},$$
   we need
   $$\frac{1}{c} = \frac{2}{e^2 - 1}.$$
   Solving for c gives
   $$\boxed{c = \frac{e^2 - 1}{2}}.$$

   (c) Geometrically, this means there is a point $x = c$ where the tangent line to the graph of $y = \ln(x)$ is parallel to the secant line through the points
   $$(1, \ln 1) = (1, 0) \quad \text{and} \quad (e^2, \ln(e^2)) = (e^2, 2).$$

3. Let $f(x) = x^3 - 3x^2 - 9x + 5$.

(a) Critical numbers occur where $f'(x) = 0$ or where $f'(x)$ is undefined. Since f is a polynomial, $f'(x)$ exists for all x. Compute

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1).$$

Thus the critical numbers are

$$\boxed{x = -1 \text{ and } x = 3}.$$

(b) Use the sign of $f'(x) = 3(x-3)(x+1)$.

x	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
$x+1$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$

So f is increasing on

$$\boxed{(-\infty, -1) \cup (3, \infty)}$$

and decreasing on

$$\boxed{(-1, 3)}.$$

(c) At $x = -1$, $f'(x)$ changes from positive to negative, so f has a local maximum at $x = -1$. At $x = 3$, $f'(x)$ changes from negative to positive, so f has a local minimum at $x = 3$. Thus:

$$\boxed{\text{local maximum at } x = -1, \qquad \text{local minimum at } x = 3.}$$

The corresponding function values are

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

and

$$f(3) = 27 - 27 - 27 + 5 = -22.$$

So the local maximum point is $(-1, 10)$ and the local minimum point is $(3, -22)$.

4. Let $f(x) = xe^{-x}$.

(a) Using the product rule,

$$f'(x) = 1 \cdot e^{-x} + x(-e^{-x}) = e^{-x} - xe^{-x} = e^{-x}(1 - x).$$

(b) Critical numbers occur where $f'(x) = 0$ or where $f'(x)$ is undefined. Since $e^{-x} > 0$ for all x, we have

$$e^{-x}(1-x) = 0 \implies 1 - x = 0 \implies x = 1.$$

So the only critical number is

$$\boxed{x = 1}.$$

(c) Because $e^{-x} > 0$ for all x, the sign of $f'(x)$ is determined by $1 - x$.

x	$(-\infty, 1)$	$(1, \infty)$
$1 - x$	$+$	$-$
$f'(x)$	$+$	$-$

Therefore, f is increasing on

$$\boxed{(-\infty, 1)}$$

and decreasing on

$$\boxed{(1, \infty)}.$$

(d) Since $f'(x)$ changes from positive to negative at $x = 1$, the First Derivative Test shows that f has a local maximum at $x = 1$.
Since

$$f(1) = 1 \cdot e^{-1} = \frac{1}{e},$$

the local maximum point is

$$\boxed{\left(1, \frac{1}{e}\right)}.$$

5. Let

$$f(x) = x^3 e^{-x^2}.$$

(a) Differentiate using the product rule:

$$f'(x) = 3x^2 e^{-x^2} + x^3 \left(-2x e^{-x^2}\right).$$

Factor:

$$f'(x) = e^{-x^2} \left(3x^2 - 2x^4\right) = x^2 e^{-x^2} (3 - 2x^2).$$

Critical numbers occur where $f'(x) = 0$. Since $e^{-x^2} > 0$ for all x, we solve

$$x^2(3 - 2x^2) = 0.$$

Thus

$$x = 0 \qquad \text{or} \qquad 3 - 2x^2 = 0.$$

So

$$x^2 = \frac{3}{2} \quad \Longrightarrow \quad x = \pm\sqrt{\frac{3}{2}}.$$

The critical numbers are

$$\boxed{x = -\sqrt{\frac{3}{2}},\ 0,\ \sqrt{\frac{3}{2}}.}$$

(b) Since

$$f'(x) = x^2 e^{-x^2} (3 - 2x^2),$$

and both x^2 and e^{-x^2} are nonnegative, the sign of $f'(x)$ is determined by $3 - 2x^2$, except that $f'(0) = 0$.

Let

$$a = \sqrt{\frac{3}{2}}.$$

Then:

x	$(-\infty, -a)$	$(-a, 0)$	$(0, a)$	(a, ∞)
$3 - 2x^2$	$-$	$+$	$+$	$-$
$f'(x)$	$-$	$+$	$+$	$-$

So f is decreasing on

$$(-\infty, -a) \quad \text{and} \quad (a, \infty),$$

and increasing on

$$(-a, 0) \quad \text{and} \quad (0, a).$$

Therefore:

- at $x = -a$, $f'(x)$ changes from negative to positive, so f has a local minimum;
- at $x = 0$, there is no sign change, so $x = 0$ is neither a local maximum nor a local minimum;
- at $x = a$, $f'(x)$ changes from positive to negative, so f has a local maximum.

Thus,

$$\boxed{x = -\sqrt{\frac{3}{2}} \text{ is a local minimum,}}$$

$$\boxed{x = 0 \text{ is neither,}}$$

$$\boxed{x = \sqrt{\frac{3}{2}} \text{ is a local maximum.}}$$

(c) Differentiate $f'(x) = x^2 e^{-x^2}(3 - 2x^2)$ to get

$$f''(x) = \left(4x^5 - 14x^3 + 6x\right) e^{-x^2}.$$

A factored form is

$$f''(x) = 2x(x^2 - 3)(2x^2 - 1)e^{-x^2}.$$

Now evaluate at the critical points.
First,

$$f''(0) = 0.$$

Next, if $x = \sqrt{\frac{3}{2}}$, then

$$f''\left(\sqrt{\frac{3}{2}}\right) < 0,$$

and if $x = -\sqrt{\frac{3}{2}}$, then

$$f''\left(-\sqrt{\frac{3}{2}}\right) > 0.$$

So we notice:

- $f''\left(-\sqrt{\frac{3}{2}}\right) > 0$, which agrees with the local minimum;
- $f''\left(\sqrt{\frac{3}{2}}\right) < 0$, which agrees with the local maximum;
- $f''(0) = 0$, so the Second Derivative Test is inconclusive at $x = 0$, which matches the fact that $x = 0$ is neither a local max nor a local min.