Calculus with Analytic Geometry II

Thomas R. Cameron

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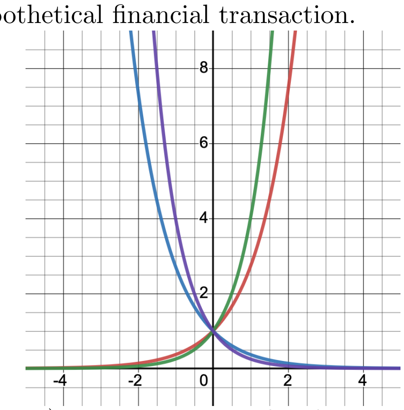
1 Exponential and Logarithmic Functions

The exponential function with base $b > 0$ is defined by $f(x) = b^x$. The long term behavior of an exponential function varies depending on the value of b. In particular, if $0 < b < 1$, then $f(x)$ approaches zero as x gets bigger; if $b = 1$, then $f(x) = 1$ for all x; if $b > 1$, then $f(x)$ approaches infinity as x gets bigger. A common choice for the base of an exponential function is Euler's number:

$$e = 2.71828\ldots.$$

This number was first introduced by Jacob Bernoulli in 1683 using a hypothetical financial transaction.

In the figure on the right, we display the exponential functions e^x (red), 4^x (green), e^{-x} (blue), and 4^{-x} (purple). Note that the graph of each of the exponential functions includes the point $(0, 1)$.



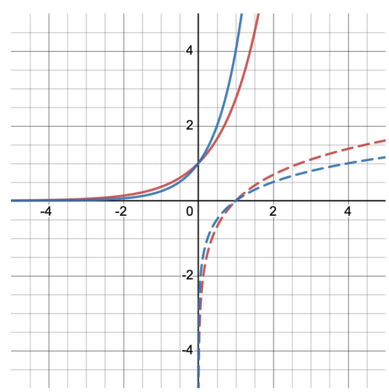
The exponential function $f(x) = b^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$. Moreover, over this domain, the exponential function $f(x)$ is one-to-one (passes the horizontal line test); hence, $f(x)$ has an inverse. By definition, the inverse function $g(x)$ satisfies

$$y = f(x) \;\Leftrightarrow\; x = g(y).$$

Hence, the inverse function $g(x)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.

In particular, the inverse of the exponential function $f(x) = b^x$ is known as the log base b function and is denoted by $g(x) = \log_b(x)$. In the case of the natural base e, the inverse function is known as the natural logarithm function and is denoted by $g(x) = \ln(x)$.

In the figure on the right, we display the exponential functions e^x (red), 4^x (blue) and their corresponding inverse functions $\ln(x)$ (dashed-red), $\log_4(x)$ (dashed-blue). Note that the graph of each of the exponential functions includes the point $(0, 1)$ and the graph of each of the logarithmic functions includes the point $(1, 0)$.



1.1 Properties

The following exponential (left) and logarithmic (right) properties are extremely useful for simplifying expressions with exponents or logs.

1. $b^0 = 1$
2. $b^x b^y = b^{x+y}$
3. $(b^x)^y = b^{xy}$
4. $\dfrac{b^x}{b^y} = b^{x-y}$

1. $\log_b(1) = 0$
2. $\log_b(xy) = \log_b(x) + \log_b(y)$
3. $\log_b\left(x^y\right) = y\log_b(x)$
4. $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

It is worth emphasizing that the log properties can be derived from the exponential properties. For example, by definition of the inverse,

$$b^0 = 1 \;\Leftrightarrow\; \log_b(1) = 0.$$

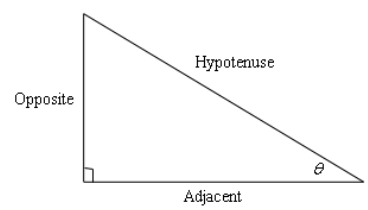
As another example, we will show that exponential property 2 implies logarithmic property 2. To this end, let $x = b^w$ and $y = b^z$. Then,

$$\begin{aligned}\log_b(xy) &= \log_b\left(b^w b^z\right)\\ &= \log_b\left(b^{w+z}\right)\\ &= w+z\\ &= \log_b\left(b^w\right) + \log_b\left(b^z\right) = \log_b(x) + \log_b(y).\end{aligned}$$

2 Trigonometric Functions

The six standard trig functions are defined over the right triangle

$$\begin{aligned}\sin(\theta) &= \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \csc(\theta) = \frac{\text{Hypotenuse}}{\text{Opposite}}\\ \cos(\theta) &= \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \sec(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}}\\ \tan(\theta) &= \frac{\text{Opposite}}{\text{Adjacent}} \qquad \cot(\theta) = \frac{\text{Adjacent}}{\text{Opposite}}\end{aligned}$$



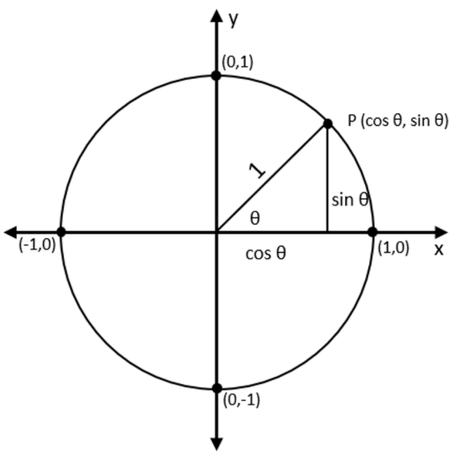
From these definitions, we can derive some important identities. To this end, let O denote the opposite side, A denote the adjacent side, and H the hypotenuse side of the right triangle. For example, consider the Pythagorean Identity:

$$\begin{aligned}\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2\\ &= \frac{O^2 + A^2}{H^2} = \frac{H^2}{H^2} = 1.\end{aligned}$$

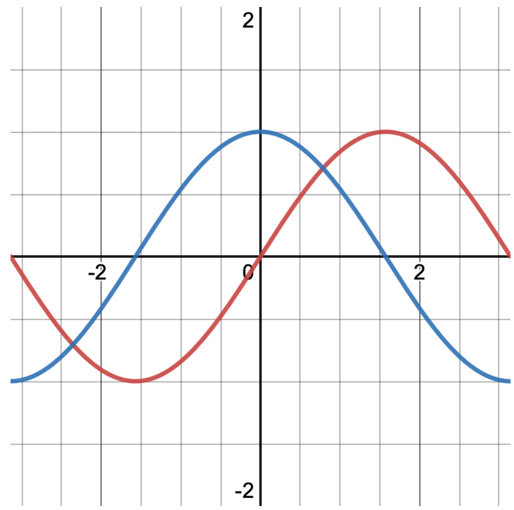
2.1 Unit Circle

By embedding right triangles into the unit circle (circle of radius 1 centered at the origin), we are able to identify points on the circle using cosine and sine functions. For example, see Figure 1a where we have a right triangle with hypotenuse 1 embedded in the 1st quadrant of the unit circle such that the adjacent side lies on the x-axis and the opposite side is parallel to the y-axis. Hence, the points on the unit circle in the 1st quadrant can be identified by $(\cos(\theta), \sin(\theta))$ as defined on the right triangle.

We extend the definitions of $\cos(\theta)$ and $\sin(\theta)$ for all angles of θ using the points on the unit circle. For example, the sine (red) and cosine (blue) functions are shown over the domain $[-\pi, \pi]$ in Figure 1b.



(a) Unit circle with right triangle embedded inside the 1st quadrant



(b) Sine (red) and Cosine (blue) functions

2.2 Inverse Trig Functions

The six standard trig functions map an angle θ to a ratio. The inverse trig functions map a ratio to an angle θ. For example,

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \Leftrightarrow \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \Leftrightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \Leftrightarrow \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Note that a function must be one-to-one on its domain in order for the inverse function to exist. Hence, we must restrict the domain of each standard trig function in order to have a well-defined inverse. For example, $\sin(x)$ is one-to-one on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, over this domain, $\sin(x)$ has a range of $[-1, 1]$. Hence, the inverse $\arcsin(x)$ is well-defined on the domain $[-1, 1]$ with range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, see Figure 2.

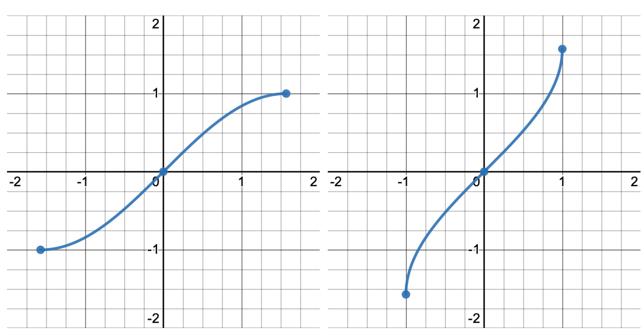


Figure 2: Sine (left) and ArcSine (right) functions