Calculus with Analytic Geometry II

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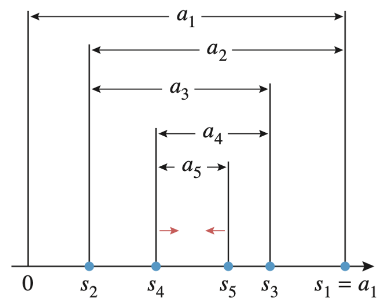
1 Alternating Series

Alternating series can be written in the following form

$$\sum_{k=1}^{\infty}(-1)^k a_k \text{ or } \sum_{k=1}^{\infty}(-1)^{k+1} a_k,$$

where the terms a_k are assumed to be positive.

An alternating series converges if $a_{k+1} \leq a_k$, for all $k \geq 1$, and $\lim_{k\to\infty} a_k = 0$. Refer to the figure on the right, note that the odd number terms form a decreasing sequence that is bounded below. Similarly, the even number terms form an increasing sequence that is bounded above. Moreover, the bound for the even and odd terms is the same; hence, the series must converge to that bound.



Note that the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges. As another example, consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}.$$

Note that

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(k+1)(k+2)} \cdot \frac{k^2+k}{k+3} = \frac{k^2+4k}{k^2+5k+6} < 1.$$

Furthermore,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{k+3}{k(k+1)} = 0.$$

Therefore, this alternating series converges.

Let S denote the value of a convergent alternate series. Then, $s_n \leq S \leq s_{n+1}$ or $s_{n+1} \leq S \leq s_n$, for all $n \geq 1$. Therefore,

$$|S - s_n| \leq a_{n+1}.$$

For example, the alternating harmonic series converges to $\ln(2)$. The 10th partial sum is

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9} - \frac{1}{10} = \frac{1627}{2520}.$$

Note that $\ln(2) - \frac{1627}{2520} \approx 0.05 < \frac{1}{11}$.

2 Absolute vs Conditional Convergence

A series $\sum_{k=1}^{\infty} u_k$ is said to converge absolutely if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k|$$

converges. If the series of absolute values diverges, then we say the original series diverges absolutely. A series is conditionaly convergent if it converges but diverges absolutely. For example, the alternating harmonic series is conditionaly convergent. As another example, consider the series alternating series $\sum_{k=0}^{\infty}(-1/2)^k$. The series of absolute values

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

is a geometric series with ratio 1/2. Hence, the give series converges absolutely.

If a series converges absolutely, then the original series also converges. Indeed, suppose that $\sum_{k=1}^{\infty} u_k$ converges absolutely. Note that

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (u_k + |u_k|) - \sum_{k=1}^{\infty} |u_k| .$$

Note that $\sum_{k=1}^{\infty} (u_k + |u_k|) \leq 2 \sum_{k=1}^{\infty} |u_k|$; hence, the direct comparison test implies that the series $\sum_{k=1}^{\infty} (u_k + |u_k|)$ converges. Therefore, $\sum_{k=1}^{\infty} u_k$ converges since it is the sum of two convergent series.

3 Exercises

Determine whether the following series converge or diverge. Classify any convergent series as absolutely or conditionally convergent.

I. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{k^2}$

II. $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{2^k}{k!}$

III. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k^k}{3^k}$