Calculus with Analytic Geometry II

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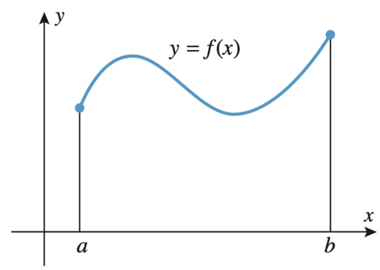
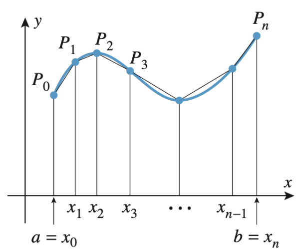
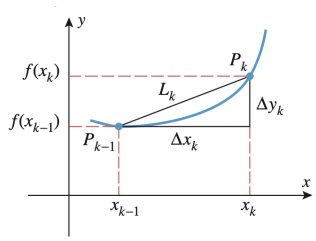
1 Arc Length

Let $f(x)$ be a smooth function on $[a, b]$, i.e., $f'(x)$ is continuous on $[a, b]$. Let L denote the arc length of $f(x)$ on $[a, b]$.



We can approximate L by dividing the interval $[a, b]$ into n subintervals $[x_{k-1}, x_k]$. Over each subinterval, we approximate the length of the curve by the line connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. We denote by L_k the length of the line segment in the kth interval. Using Pythagorean's theorem, we find





that

$$
\begin{aligned}
L_k &= \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2} \\
&= \Delta x_k \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2} \\
&= \Delta x_k \sqrt{1 + (f'(x_k^*))^2},
\end{aligned}
$$

where the last equation follows from the mean value theorem. Therefore, we can approximate the arc length as follows:

$$L \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \Delta x_i.$$

Taking the limit as $n \to \infty$ gives us

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \Delta x_i = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

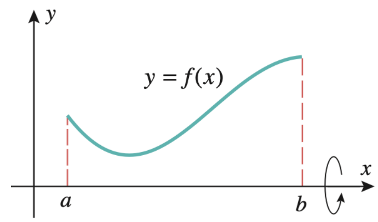
Note that, if the curve is expressed as $y = f(x)$, then the arc length formula can be written as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

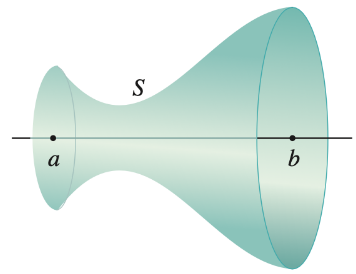
Similarly, if the curve is expressed as $x = f(y)$, then the arc length formula can be written as

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

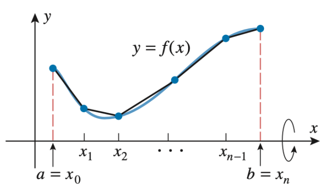
2 Surface of Revolution

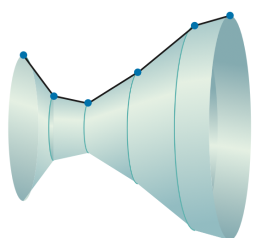


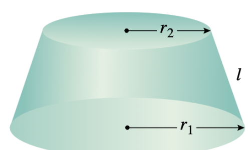
Let $f(x)$ denote a smooth function on $[a, b]$. Consider the surface of revolution formed by revolving the curve about the x-axis, and let S denote the surface area of this surface of revolution.



To approximate S we split the interval $[a, b]$ into n subintervals. Over each subinterval $[x_{k-1}, x_k]$, we approximate the curve $y = f(x)$ via the line segment connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. When these line segments are revolved about the x-axis, it generates a surface consisting of n parts, each of which is a portion of a right circular cone.







The area of each approximating surface is given by

$$S_k = 2\pi \left(\frac{f(x_{k-1}) + f(x_k)}{2}\right) L_k,$$

where L_k denotes the length of the line segment connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. Recall that

$$L_k = \Delta x_k \sqrt{1 + f'(x_k^*)^2},$$

where x_k^* is in $[x_{k-1}, x_k]$ such that $f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$, guaranteed by the mean value theorem. Furthermore, since $f(x)$ is continuous, the intermediate value theorem implies that there exists a x_k^{**} in $[x_{k-1}, x_k]$ such that

$$\frac{f(x_{k-1}) + f(x_k)}{2} = f(x_k^{**}).$$

Therefore, the surface area S can be approximated via

$$S \approx 2\pi \sum_{i=1}^{n} f(x_i^{**})\sqrt{1 + f'(x_i^*)^2}\Delta x_i.$$

Taking the limit as $n \to \infty$ gives us

$$S = \lim_{n\to\infty} 2\pi \sum_{i=1}^{n} f(x_i^{**})\sqrt{1 + f'(x_i^{*})^2}\Delta x_i = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}dx.$$