Calculus with Analytic Geometry II

Thomas R. Cameron

January 22, 2025

1 Area of a Region Between Two Curves

Let $f(x)$ and $g(x)$ denote continuous functions on $[a, b]$ such that

$$f(x) \geq g(x),\ a \leq x \leq b.$$

We are interested in the area of the region bounded above by $y = f(x)$ and below by $y = g(x)$, over the interval $[a, b]$, we denote this area by A. Note that in this area problem, the curves $y = f(x)$ and $y = g(x)$ are allowed to touch, but not cross.

To solve this problem, we partition the interval $[a, b]$ into $n \geq 1$ subintervals each of length Δx. Then, for each subinterval $[x_{i-1}, x_i]$, we select a point c_i, so that the area of the region can be approximated as

$$A \approx \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x.$$

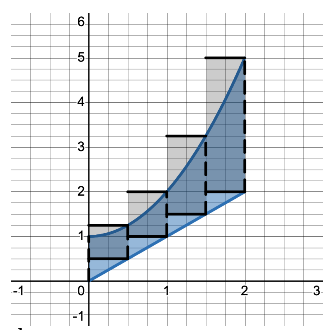
Taking $n \to \infty$ gives an exact formula for the area

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx.$$

As an example, consider the functions

$$f(x) = 1 + x^2 \geq x = g(x),\ 0 \leq x \leq 2.$$

Let A denote the area of the region bounded above by $y = 1 + x^2$ and below by $y = x$, over the interval $[0, 2]$.



An approximation of A is shown in the figure on the right, where the number of subintervals is $n = 4$. Note that

$$\begin{aligned} A &\approx \frac{1}{2}\left[\left(\frac{5}{4}\right) - \left(\frac{1}{2}\right)\right] + \frac{1}{2}[(2) - (1)] + \frac{1}{2}\left[\left(\frac{13}{4}\right) - \left(\frac{3}{2}\right)\right] + \frac{1}{2}[(5) - (2)] \\ &= \frac{1}{2}\left(\frac{3}{4} + 1 + \frac{7}{4} + 3\right) = \frac{1}{2}\left(\frac{26}{4}\right) = \frac{13}{4}. \end{aligned}$$

We can make this approximation exact by taking $n \to \infty$. To this end, note that

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i^2}{n^2}\right) - \left(\frac{2i}{n}\right)\right] \frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2}{n}\left(n + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\frac{n(n+1)}{2}\right) \\ &= 2 + \frac{16}{6} - \frac{4}{2} = \frac{8}{3}. \end{aligned}$$

We can also evaluate this area problem directly using the Fundamental Theorem of Calculus:

$$\begin{aligned} A &= \int_0^2 \left[(1+x^2) - (x)\right] dx \\ &= \left(x + \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^2 = \frac{8}{3}. \end{aligned}$$

2 Area of a Region Between Intersecting Curves

Let $f(x)$ and $g(x)$ denote continuous functions that intersect to form a closed region. We are interested in the area of this closed region, which we denote by A. To calculate A, we find the points of intersection, denoted $x_1 < x_2 < \cdots < x_k$, $k \geq 2$, and compute the area between two curves over each interval $[x_i, x_{i+1}]$, $1 \leq i \leq k-1$.

As an example, consider the functions

$$f(x) = 2 - x^2, \;\; g(x) = x.$$

The points of intersection can be found by solving $f(x) = g(x)$, i.e.,

$$x^2 + x - 2 = 0 \;\Rightarrow\; (x+2)(x-1) = 0,$$

hence $x_1 = -2$ and $x_2 = 1$. Note that, over the interval $[-2, 1]$ $f(x) \geq g(X)$. Therefore, the area of the closed region is given by

$$\begin{aligned} A &= \int_{-2}^1 \left[(2-x^2) - (x)\right] dx \\ &= \left(2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{-2}^1 = \frac{9}{2}. \end{aligned}$$

Next, consider the functions

$$f(x) = 3x^3 - x^2 - 10x, \;\; g(x) = -x^2 + 2x.$$

The points of intersection can be found by solving $f(x) = g(x)$, i.e.,

$$3x^3 - 12x = 3x(x^2 - 4) = 0,$$

hence $x_1 = -2$, $x_2 = 0$, $x_3 = +2$. Note that, over the interval $[-2, 0]$ $f(x) \geq g(x)$, and over the interval $[0, 2]$ $g(x) \geq f(x)$. Therefore, the area of the closed region is given by

$$\begin{aligned} A &= \int_{-2}^0 \left[(3x^3 - x^2 - 10x) - (-x^2 + 2x)\right] dx + \int_0^2 \left[(-x^2 + 2x) - (3x^3 - x^2 - 10x)\right] dx \\ &= \left(\frac{3}{4}x^4 - 6x^2\right)\Big|_{-2}^0 + \left(6x^2 - \frac{3}{4}x^4\right)\Big|_0^2 \\ &= 12 + 12 = 24. \end{aligned}$$