Calculus with Analytic Geometry II

Thomas R. Cameron

February 24, 2025

1 The Comparison Test

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where a_k and b_k are non-negative and satisfy

$$a_k \leq b_k,\ k \geq 1.$$

Then, the following statements hold

I. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

II. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ also diverges.

As an example, consider the following series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - 1/2}.$$

Note that $\sqrt{k} - 1/2 < \sqrt{k}$ for all $k \geq 1$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - 1/2} > \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ divergets, it follows that the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-1/2}$ also diverges.

As another example, consider the following series

$$\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}.$$

Note that $2k^2 + k > k^2$ for all $k \geq 1$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{2k^2 + k} < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, it follows that $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ also converges.

2 The Limit Comparison Test

Sometimes it is difficult to find a series that is useful for direct comparison. In this case, the limit comparison test may be easier to apply. Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where a_k and b_k are positive and the following limit

$$\lim_{k \to \infty} \frac{a_k}{b_k}$$

is finite and non-zero. Then, the series either both converge or both diverge. A proof of the limit comparison test follows.

Proof. Suppose that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L,$$

where $L > 0$ is finite. Then, there exists a N such that

$$L - 1 < \frac{a_k}{b_k} < L + 1,$$

for all $k > N$. Hence,

$$\sum_{k=N+1}^{\infty} (L-1)b_k < \sum_{k=N+1}^{\infty} a_k < \sum_{k=N+1}^{\infty} (L+1)b_k.$$

□

3 Ratio and Root Tests

Let $\sum_{k=1}^{\infty} u_k$ be a series, where u_k are positive for all $k \geq 1$. The ratio test defines

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}.$$

Then,

a. if $\rho < 1$, then the series converges.

b. if $\rho > 1$, then the series diverges.

c. if $\rho = 1$, then the ratio test is inclunclusive.

The root test defines

$$\rho = \lim_{k\to\infty} u_k^{1/k}.$$

Then,

a. if $\rho < 1$, then the series converges.

b. if $\rho > 1$, then the series diverges.

c. if $\rho = 1$, then the root test is inclunclusive.

4 Exercises

Use the limit comparison test to determine the convergence/divergence of

I. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$

II. $\displaystyle\sum_{k=1}^{\infty} \frac{2k^3 - 7k}{k^4 + 3k^2 + 1}$

Use the ratio test to determine the convergence/divergence of

a. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

b. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$

Use the root test to determine the convergence/divergence of

a. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{4k-1}{2k+1} \right)^k$

b. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\ln(k+1)^k}$