Calculus with Analytic Geometry II

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April 2, 2025

1 Conic Sections

Circles, ellipses, parabolas, and hyperbolas are called conic sections or conics because they can be obtained as intersections of a plane with a double-napped circular cone, see Figure 1.

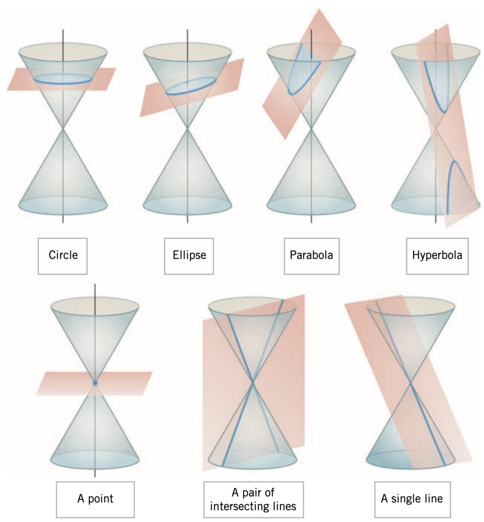

Figure 1: Conic sections for circles, ellipses, parabolas, and hyperbolas.

It is better suited for calculus to define these conic sections based on their geometric properities. For example, a parabola is the set of points in the plane that are equidistant from a fixed line and a fixed point not on the line, see Figure 2 (left). It is traditional to denote the distance from the vertex to the focus as p,

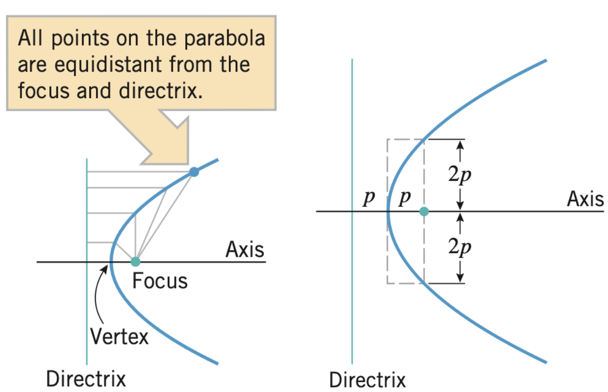

Figure 2: Parabola geometric description.

see Figure 2 (right). Then, we can use the geometric description of a parabola to determine an equation for all points (x, y) on the parabola. For example, with vertex (h, k) and directix $y = k - p$, we have

$$\begin{aligned}(x-h)^2 &= \left(y-(k-p)\right)^2 - \left(y-(k+p)\right)^2 \\ &= \left(y^2 - 2y(k-p) + (k-p)^2\right) - \left(y^2 - 2y(k+p) + (k+p)^2\right) \\ &= 4p(y-k).\end{aligned}$$

An ellipse is the set of all points in the plane, the sum of whose distances from two fixed points is a given positive constant that is greater than the distance between the fixed points, see Figure 3 (left). It is

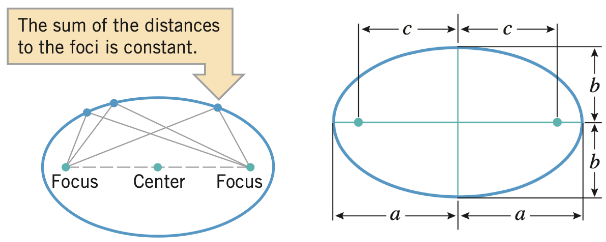

Figure 3: Ellipse geometric description.

traditional to denote the semimajor axis by a, the semiminor axis by b, and the distance from the center to the focus by c, see Figure 3 (right). There is a basic relationship between the numbers a, b, and c that can be obtained by examining the sum of the distances to the foci from a point P at the end of the major axis and from a point Q at the end of the minor axis. In particular,

$$a = \sqrt{b^2 + c^2} \text{ or } c = \sqrt{a^2 - b^2}.$$

Then, we can derive the general formula of an ellipse with center (h, k) and major axis $y = k$:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

With center (h, k) and major axis $x = h$, the equation of the ellipse:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

2 Exercises

1. Find the vertex and focus of the parabola $y = \frac{1}{2} - x - \frac{1}{2}x^2$.
2. Find the center, foci, and eccentricity of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.