Calculus with Analytic Geometry II

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## 1 Review

Recall that conic sections, such as circles, ellipses, parabolas, and hyperbolas can be viewed obtained from the intersection of a half plane and and double napped-circular cone. However, it is better suited to derive their mathematical formulations from their geometric descriptions. For example, the points on a parabola are equidistant from focus and the directix. It is traditional to denote the distance from the vertex to the foci by $p$. In this case, the point $(x, y)$ on the parabola with vertex $(h, k)$ and directix $y = k - p$ is given by

$$(x-h)^2 = 4p(y-k).$$

The points on a ellipse have a sum of distances to the foci that is constant. It is traditional to denote the distance from the center to the foci by $c$, the length of the major axis by $a$, and the minor axis by $c$. In this case, we have $a = \sqrt{b^2 + c^2}$ or $c = \sqrt{a^2 - b^2}$ Moreover, the point $(x, y)$ on the ellipse with center $(h, k)$ and foci $(h - c, k)$ and $(h + c, k)$ is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

## 2 Hyperbola

The points on a hyperbola have distances from two fixed point whose difference is a positive constant that is less than the distance between the two fixed points, see Figure 1 (left). In Figure 1 (middle), it is shown

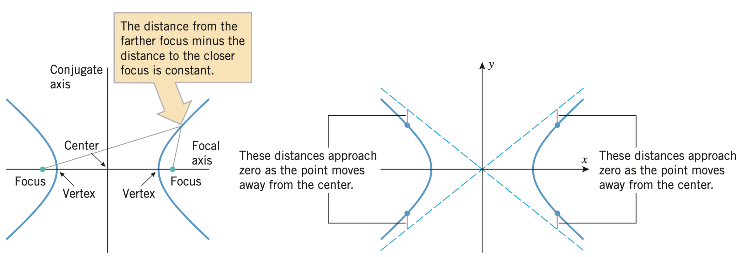


Figure 1: Hyperbola geometric description

that the distance from each point on the hyperbola approaches an oblique asymptote as they move away from the center.

It is traditional to denote the distance between the two vertices by $2a$ and the distance between the foci by $2c$, see Figure 2 (left). Moreover, we define the quantity $b = \sqrt{c^2 - a^2}$, which is pictured geometrically in Figure 2 (right). For a given vertex, the distance to the closer foci is $(c - a)$ and the distance to the farther foci is $2a + (c - a)$. Hence, the difference is given by $2a$. Therefore, for all points on the parabola, the distance from the farther foci minus the distance to the closer foci is $2a$. So, the point $(x, y)$ on the hyperbola with center $(h, k)$ and foci $(h - c, k)$ and $(h + c, k)$ is given by

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

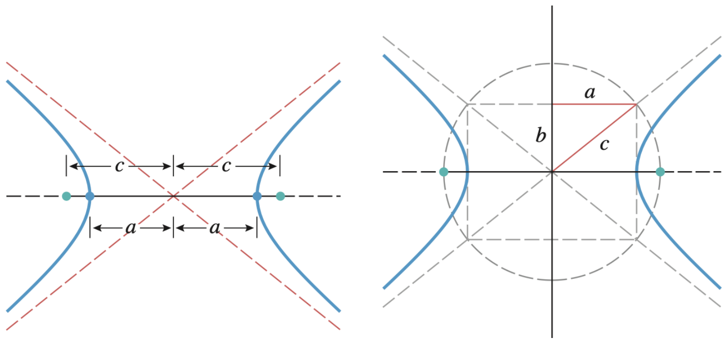


Figure 2: Hyperbola mathematical description

The equation for the hyperbola can be re-written as follows

$$(y-k) = \pm \frac{b}{a}\sqrt{(x-h)^2 - a^2}$$

Therefore, as $x \to \infty$, the points on the hyperbola approach the lines

$$y = k \pm \frac{b}{a}(x-h),$$

which define the oblique asymptotes of the hyperbola.

# 3 Eccentricity

The eccentricity of a conic section is the ratio of the distance from points along the curve to the directrix to the distance from points along the curve to the foci. We denote the eccentricity by $e$ (sorry Euler's number), see Figure 3.

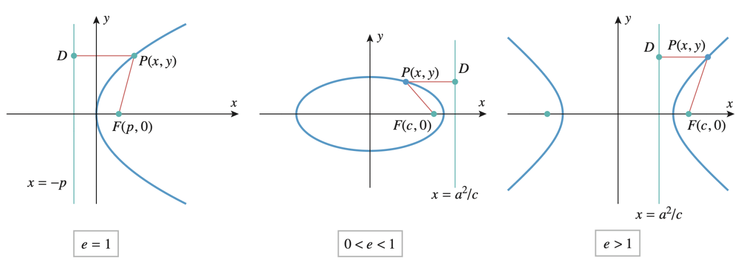


Figure 3: Eccentricity of conic sections

# 4 Exercises

1. Find the center, foci, and eccentricity of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.
2. Find the center, foci, and eccentricity of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$