Calculus with Analytic Geometry II

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January 16, 2025

1 Limit Definition

The derivative of the function $f(x)$ is defined by

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. For example, consider $f(x) = x^2$. Then,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) = 2x.
\end{aligned}$$

As a second example, consider $f(x) = \sqrt{x+5}$. Then,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+5+h} - \sqrt{x+5}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+5+h} - \sqrt{x+5}}{h} \cdot \frac{\sqrt{x+5+h} + \sqrt{x+5}}{\sqrt{x+5+h} + \sqrt{x+5}} \\
&= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+5+h} + \sqrt{x+5}\right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+5+h} + \sqrt{x+5}} = \frac{1}{2\sqrt{x+5}}.
\end{aligned}$$

2 Derivative Rules

The limit definition of the derivative is used to derive rules for evaluating derivatives.

2.1 Basic Rules

Let $f(x)$ and $g(x)$ denote functions. Then, the derivative of $f(x) + g(x)$ can be derived as follows:

$$\begin{aligned}
\frac{d}{dx}\left(f(x) + g(x)\right) &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

Similarly, rules for the product, quotient, and composition of functions may be derived. We summarize these rules below:

$$\frac{d}{dx}c = 0, \quad \frac{d}{dx}cf(x) = cf'(x)$$
$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x), \quad \frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x)+f(x)g'(x)$$
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}, \quad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$
$$\frac{d}{dx}x^n = nx^{n-1}$$

We use these basic rules to differentiate more complex functions. For example,

$$\begin{aligned}\frac{d}{dx}x\sqrt{x^2+3x+5} &= \sqrt{x^2+3x+5}\frac{d}{dx}x + x\frac{d}{dx}\sqrt{x^2+3x+5}\\ &= \sqrt{x^2+3x+5} + x\frac{1}{2}\left(x^2+3x+5\right)^{-1/2}(2x+3)\\ &= \sqrt{x^2+3x+5} + \frac{2x^2+3x}{2\sqrt{x^2+3x+5}}.\end{aligned}$$

2.2 Transcendental Rules

Using the squeeze theorem, one can show that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1, \ \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$$

Therefore, we have

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h} = \cos(x).\end{aligned}$$

We summarize the rules for several transcendental functions below:

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

We can use the above rules to derive derivative rules for other transcendental functions. For example,

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).\end{aligned}$$

2.3 Implicit Differentiation

When we write $y = f(x)$, we have written the variable y explicitly as a function of x. However, not all mathematical expressions can be solved explicitly for a single variable. For example, the equation of a circle with radius 1 centered at the origin is given by

$$x^2+y^2 = 1,$$

which is called an implicit equation since their is an implied relationship between the variables x and y. When differentiating implicit equations, we apply d/dx to both sides of the equation. For example,

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}1\\ \frac{d}{dx}x^2+\frac{d}{dx}y^2 &= 0\\ 2x+2y\frac{dy}{dx} &= 0\end{aligned}$$

Note the dy/dx term in the last line above, which result from the chain rule when taking the derivative of y^2 with respect to x. We can now solve for dy/dx as follows:

$$\frac{dy}{dx}=-\frac{x}{y}.$$

We can use implicit differentiation to solve for the derivative of inverse functions. Suppose $f(x)$ and $g(x)$ are inverse functions and we wish to compute the derivative of $g(x)$. If we let $y = g(x)$, then $f(y) = x$. Applying d/dx to both sides gives us

$$\begin{aligned}\frac{d}{dx}f(y) &= \frac{d}{dx}x\\ f'(y)\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \frac{1}{f'(y)}=\frac{1}{f'(g(x))}\end{aligned}$$

As an example, we will compute the derivative of $y = \arccos(x)$. Note that, $\cos(y) = x$. Applying d/dx to both sides gives us

$$\begin{aligned}\frac{d}{dx}\cos(y) &= \frac{d}{dx}x\\ -\sin(y)\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= -\frac{1}{\sin(y)}=-\frac{1}{\sin(\arccos(x))}.\end{aligned}$$

The above formula can be simplified by considering a right triangle with angle y, adjacent side x and hypotenuse 1. For such a triangle, $\sin(y) = \sqrt{1-x^2}$. Hence,

$$\frac{d}{dx}\arccos(x)=-\frac{1}{\sqrt{1-x^2}}.$$