Calculus with Analytic Geometry II

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May 20, 2025

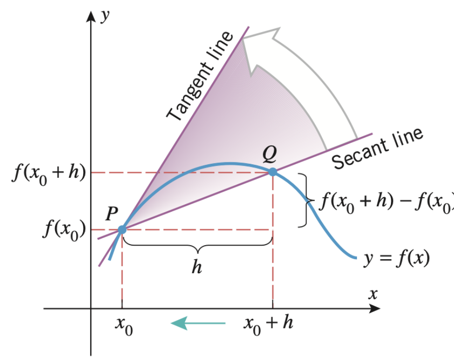
1 Limit Definition

The derivative of the function $f(x)$ is defined by

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists and is finite.

The limit definition of the derivative is derived from the slope of the tangent line, see figure on the right. In particular, the slope of the tangent line at point P is taken as the limit of the secant line connecting points P and Q as the point Q approaches the point P.



For example, consider $f(x) = x^2$. Then,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{(x^2+2xh+h^2)-x^2}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2}{h}\\
&= \lim_{h\to 0}(2x+h) = 2x.
\end{aligned}$$

As a second example, consider $f(x) = \sqrt{x+5}$. Then,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{\sqrt{x+5+h}-\sqrt{x+5}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+5+h}-\sqrt{x+5}}{h}\cdot\frac{\sqrt{x+5+h}+\sqrt{x+5}}{\sqrt{x+5+h}+\sqrt{x+5}}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+5+h}+\sqrt{x+5}\right)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+5+h}+\sqrt{x+5}} = \frac{1}{2\sqrt{x+5}}.
\end{aligned}$$

2 Derivative Rules

The limit definition of the derivative is used to derive rules for evaluating derivatives.

2.1 Basic Rules

Let $f(x)$ and $g(x)$ denote functions. Then, the derivative of $f(x) + g(x)$ can be derived as follows:

$$\begin{aligned}
\frac{d}{dx}\left(f(x) + g(x)\right) &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

Similarly, rules for the product, quotient, and composition of functions may be derived. We summarize these rules below:

$$\begin{gathered}
\frac{d}{dx} c = 0, \quad \frac{d}{dx} c f(x) = c f'(x) \\
\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x), \quad \frac{d}{dx}\left(f(x) g(x)\right) = f'(x) g(x) + f(x) g'(x) \\
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}, \quad \frac{d}{dx} f(g(x)) = f'(g(x)) g'(x) \\
\frac{d}{dx} x^n = n x^{n-1}
\end{gathered}$$

We use these basic rules to differentiate more complex functions. For example,

$$\begin{aligned}
\frac{d}{dx} x\sqrt{x^2 + 3x + 5} &= \sqrt{x^2 + 3x + 5} \frac{d}{dx} x + x \frac{d}{dx} \sqrt{x^2 + 3x + 5} \\
&= \sqrt{x^2 + 3x + 5} + x \frac{1}{2}\left(x^2 + 3x + 5\right)^{-1/2} (2x + 3) \\
&= \sqrt{x^2 + 3x + 5} + \frac{2x^2 + 3x}{2\sqrt{x^2 + 3x + 5}}.
\end{aligned}$$

2.2 Transcendental Rules

Using the squeeze theorem, one can show that

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1, \ \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.$$

Therefore, we have

$$\begin{aligned}
\frac{d}{dx} \sin(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \sin(x) \frac{\cos(h) - 1}{h} + \lim_{h \to 0} \cos(x) \frac{\sin(h)}{h} = \cos(x).
\end{aligned}$$

We summarize the rules for several transcendental functions below:

$$\begin{gathered}
\frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x) \\
\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \ln(x) = \frac{1}{x}
\end{gathered}$$

We can use the above rules to derive derivative rules for other transcendental functions. For example,

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).\end{aligned}$$

2.3 Implicit Differentiation

When we write $y = f(x)$, we have written the variable y explicitly as a function of x. However, not all mathematical expressions can be solved explicitly for a single variable. For example, the equation of a circle with radius 1 centered at the origin is given by

$$x^2 + y^2 = 1,$$

which is called an implicit equation since their is an implied relationship between the variables x and y. When differentiating implicit equations, we apply d/dx to both sides of the equation. For example,

$$\begin{aligned}\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}1 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0\end{aligned}$$

Note the dy/dx term in the last line above, which result from the chain rule when taking the derivative of y^2 with respect to x. We can now solve for dy/dx as follows:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

We can use implicit differentiation to solve for the derivative of inverse functions. Suppose $f(x)$ and $g(x)$ are inverse functions and we wish to compute the derivative of $g(x)$. If we let $y = g(x)$, then $f(y) = x$. Applying d/dx to both sides gives us

$$\begin{aligned}\frac{d}{dx}f(y) &= \frac{d}{dx}x \\ f'(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{f'(y)} = \frac{1}{f'(g(x))}\end{aligned}$$

As an example, we will compute the derivative of $y = \arccos(x)$. Note that, $\cos(y) = x$. Applying d/dx to both sides gives us

$$\begin{aligned}\frac{d}{dx}\cos(y) &= \frac{d}{dx}x \\ -\sin(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin(y)} = -\frac{1}{\sin(\arccos(x))}.\end{aligned}$$

The above formula can be simplified by considering a right triangle with angle y, adjacent side x and hypotenuse 1. For such a triangle, $\sin(y) = \sqrt{1-x^2}$. Hence,

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

3 Antiderivatives

Let $f(x)$ be a function. Then, $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$. Note that, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. The indefinite integral of $f(x)$, denoted $\int f(x)dx$, represents the family of antiderivatives of $f(x)$. In particular,

$$\int f(x)dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant. Note that the differential dx determines that $F'(x) = \frac{d}{dx}F(x)$, that is, we are differentiating $F(x)$ with respect to x.

We can derive many basic integration rules simply from our knowledge of derivatives. Below is a summary of several integration rules:

$$\begin{aligned}
\frac{d}{dx}C = 0 \;&\Rightarrow\; \int 0dx = C \\
\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x) \;&\Rightarrow\; \int \left(f(x) + g(x)\right)dx = \int f(x)dx + \int g(x)dx \\
\frac{d}{dx}x^n = nx^{n-1} \;&\Rightarrow\; \int x^n dx = \frac{x^{n+1}}{n+1} + C \\
\frac{d}{dx}\sin(x) = \cos(x) \;&\Rightarrow\; \int \cos(x)dx = \sin(x) + C \\
\frac{d}{dx}e^x = e^x \;&\Rightarrow\; \int e^x = e^x + C \\
\frac{d}{dx}\ln(x) = \frac{1}{x} \;&\Rightarrow\; \int \frac{1}{x}dx = \ln|x| + C
\end{aligned}$$

4 Area Bounded by Curves

Let $f(x)$ be a continuous function on the interval $[a, b]$. The definite integral of $f(x)$ over $[a, b]$, denoted $\int_a^b f(x)dx$, is the signed area of the region bounded by the curve $f(x)$ and the x-axis over the interval $[a, b]$. By signed area, we mean that any portion of the region above the x-axis has positive area, while any portion below the x-axis has negative area.

For simple regions, we can evaluate the definite integral using geometric formulas for area. For example,

$$\int_{0,2} (2 - x)dx$$

represents the area of the triangle bounded by the x-axis, y-axis, and the line $y = 2 - x$. The area of this triangle is 2; hence, $\int_0^2 (2 - x)dx = 2$.

For more complicated regions, we use the limit definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i,$$

where the interval $[a, b]$ is partitioned into n subintervals $[x_{i-1}, x_i]$, $1 \le i \le n$, the length of the ith subinterval is Δx_i, and c_i is any point in the subinterval $[x_{i-1}, x_i]$. As an example, consider the definite integral $\int_0^1 x^2dx$. We partition the itnerval $[0, 1]$ into n subitnervals $[x_{i-1}, x_i] = [\frac{i-1}{n}, \frac{i}{n}]$, and we select $c_i = \frac{i}{n}$ for each

$1 \leq i \leq n$. Hence,

$$\begin{aligned}\int_0^1 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2\\ &= \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{1}{6} n(n+1)(2n+1)\right) = \frac{1}{3}.\end{aligned}$$

5 Fundamental Theorem of Calculus

One of the most impportant theorems in all of calculus describes the relationship between the definite and indefinite integrals. In particular, if $f(x)$ is a continuous function on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Hence, the complex problem of determining the area of any region can be solved by finding an antiderivative of a function. For Example,

$$\int_0^{\pi/2} \cos(x)dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1.$$

6 U-Substitution

The one method for integration covered in Calc I is U-Substitution. In this method, we seek a change of variables to transform a complex integral to a simple integral. U-substitution is particularly useful when the given integration problem can be viewed in the following form:

$$\int f(g(x))g'(x)dx.$$

Then, we make the substitution $u = g(x)$ so that $du = g'(x)dx$ and our integral becomes

$$\int f(u)du.$$

For example,

$$\begin{aligned}\int \cos(x)e^{\sin(x)}dx &= \int e^u du, \qquad u = \sin(x)\\ &= e^u + C = e^{\sin(x)} + C.\end{aligned}$$

Some substitutions are less apparent. For example, consider the following integral:

$$\int x^2\sqrt{x-1}dx.$$

We let $u = x - 1$, so that $du = dx$ and $x^2 = (u+1)^2$. Then, we have

$$\begin{aligned}
\int x^2\sqrt{x-1}dx &= \int (u+1)^2\sqrt{u}du \\
&= \int (u^2+2u+1)u^{1/2}du \\
&= \int u^{5/2} + 2u^{3/2} + u^{1/2}du \\
&= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.
\end{aligned}$$

Finally, consider the following integral:

$$\int \frac{1}{4+x^2}dx = \frac{1}{4}\int \frac{1}{1+(x/2)^2}dx.$$

We let $u = x/2$, so that $du = dx/2$. Then, we have

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{1+(x/2)^2}dx &= \frac{1}{4}\int \frac{1}{1+u^2}2du \\
&= \frac{1}{2}\int \frac{1}{1+u^2}du \\
&= \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$