Calculus with Analytic Geometry II

Thomas R. Cameron

June 4, 2025

1 Geometric Series

A geometric series can be written as

$$\sum_{k=0}^{\infty} ar^k,$$

where a and r are non-zero. If $|r| \geq 1$, then the series diverges.

Now, consider the partial sum

$$s_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n.$$

Note that $s_n - rs_n = a - ar^{n+1}$. Therefore,

$$s_n = \frac{a - ar^{n+1}}{1-r}.$$

Hence, if $|r| < 1$, then the geometric series converges:

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

As an example, consider the series $\sum_{k=0}^{\infty} 3 \cdot 5^{-k}$, which is a geometric series with $a = 3$ and $r = 1/5$. Therefore,

$$\sum_{k=0}^{\infty} 3 \cdot 5^{-k} = \frac{3}{1 - 1/5}.$$

As another example, consider the series $\sum_{k=2}^{\infty} 5 \cdot 3^{-k}$, which is a geometric series with $a = 5/9$ and $r = 1/3$. We can perform an index shit on this series as follows:

$$\sum_{k=2}^{\infty} 5 \cdot 3^{-k} = \sum_{k=0} \frac{5}{3^2} \cdot \frac{1}{3^k}.$$

2 Telescoping Series

A telescoping series can be written in the form

$$\sum_{k=1}^{\infty} (a_k - a_{k-1}).$$

Hence, the partial sum is given by

$$s_n = \sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0,$$

for all $n \geq 1$. If $\lim_{n\to\infty} a_n = L$, then

$$\sum_{k=1}^{\infty} (a_k - a_{k-1}) = \lim_{n\to\infty} s_n = L - a_0.$$

As an example, consider the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. The partial sum is given by

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1},$$

for all $n \geq 1$. Hence,

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

As another example, consider the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$. Applying partial fraction decomposition, we find that

$$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1/2}{k-1} + \frac{-1/2}{k+1}.$$

So the series can be written as

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \sum_{k=2}^{\infty} \frac{1}{2} \left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

Note that

$$\begin{aligned}
\sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1}\right) &= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1}\right) \\
&= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) + \sum_{k=2}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(1 - \frac{1}{n}\right) + \left(\frac{1}{2} - \frac{1}{n+1}\right).
\end{aligned}$$

Thereofre,

$$\frac{1/2}{k-1} + \frac{-1/2}{k+1} = \frac{1}{2} \lim_{n\to\infty} \left(1 - \frac{1}{n} + \frac{1}{2} - \frac{1}{n+1}\right) = \frac{3}{4}.$$