Calculus with Analytic Geometry II

Thomas R. Cameron

March 24, 2025

1 Power Series Review

1.1 Taylor Series

Let $f(x)$ be infinitely differentiable at x_0. For a nonnegative integer n, the nth degree Taylor polynomial is

$$p_n(x) = f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + \cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}. \tag{1}$$

Using integration by parts, we derived the integral remainder theorem: For any x,

$$f(x) = p_n(x) + \int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt \tag{2}$$

and the remainder is bounded above by

$$\left|\int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt\right| \leq M\frac{|x-x_0|^{n+1}}{(n+1)!}, \tag{3}$$

where M is the maximum value $\left|f^{(n+1)}(t)\right|$ for all t between x_0 and x.

The Taylor series is

$$\lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}, \tag{4}$$

which is a power series centered at x_0 with coefficients $c_k = \frac{f^{(k)}(x_0)}{k!}$. Given x, this Taylor series converges to $f(x)$ if and only if the Taylor polynomial remainder goes to zero as $n \to \infty$.

1.2 Interval of Convergence

Given a power series $\sum_{k=0}^{\infty} c_k(x-x_0)^k$, one of the following is true

a. The series only converges for $x = x_0$.

b. The series converges absolutely for all x.

c. The series converges absolutely for all x in the interval $(x_0 - R, x_0 + R)$ and diverges for $|x - x_0| > R$. For $|x - x_0| = R$, the series may converge or diverge.

The values of x for which the power series converges is known as the interval of convergence. For the power series generated as the Taylor series of $f(x)$ at x_0, we are interested in its interval of convergence and whether the series converges to $f(x)$. If the power series converges to $f(x)$ for all x in a open interval, then we say that $f(x)$ is analytic on that interval. Recall, we've seen examples of Taylor series that converge but don't converge to $f(x)$.

1.3 Power Series Representation of functions

a. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for all $-\infty < x < \infty$.

b. $\ln(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$, for all $0 < x \leq 2$.

c. $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, for all $-1 < x < 1$.

d. $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, for all $-\infty < x < \infty$.

e. $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$, for all $-\infty < x < \infty$.

1.4 Operations on Power Series

We can perform algebraic and calculus based operations on power series to combine series, change the interval of convergence, and create new series. For example, using series (b.) we find

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \qquad -1 < x \leq 1$$

and

$$\ln(1-x) = \sum_{k=1}^{\infty} -\frac{x^k}{k}, \qquad -1 \leq x < 1.$$

Using logarithm properties, we obtain

$$\begin{aligned} \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} + \sum_{k=1}^{\infty} \frac{x^k}{k} \\ &= \sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1}, \qquad -1 < x < 1. \end{aligned}$$

As another example, using series (c.) we find

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad -1 < x < 1$$

and

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \ -1 < x < 1.$$

Therefore,

$$\begin{aligned} \arctan(x) + C &= \int \frac{1}{1+x^2} dx \\ &= \sum_{k=0}^{\infty} (-1)^k \int x^{2k} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots . \end{aligned}$$