Calculus with Analytic Geometry II

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1 Definite Integrals

Let $f(x)$ be a continuous function and let $a < b$. Then, the definite integral $\int_a^b f(x)dx$ represents the signed area of the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$. This area can be computed via Riemann sums:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where the interval $[a, b]$ is split into $n \geq 1$ subintervals, each of length $\Delta x = \frac{b-a}{n}$, and c_i denotes any point in the subinterval $[x_{i-1}, x_i]$ for $1 \leq i \leq n$.

2 Improper Integrals

Our goal is to extend the definition of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We refer to such integrals as improper. Here are some examples of improper integrals with infinite intervals of integration:

$$\int_1^\infty \frac{dx}{x^2},\ \int_{-\infty}^0 e^x dx,\ \int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

Here are some examples of improper integrals with integrands that have vertical asymptotes within the interval of integration:

$$\int_{-3}^3 \frac{dx}{x^2},\ \int_1^2 \frac{dx}{x-1},\ \int_0^\pi \tan(x)dx.$$

Here are some examples of improper integrals with infinite intervals of integration and integrands that have vertical asymptotes within the interval of integration:

$$\int_0^\infty \frac{dx}{\sqrt{x}},\ \int_{-\infty}^\infty \frac{dx}{x^2-9},\ \int_1^\infty \sec(x)dx.$$

The formal definition of an improper integral is given by the limit of definite integrals. We begin with continuous functions over infinite intervals of integration. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

If $f(x)$ is continous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)dx,$$

where c is real value. In each case, if the limit exists, then the improper integral is said to converge. Otherwise, the improper integral is said to diverge.

Next, we consider functions with vertical asymptotes within the interval of integration. If $f(x)$ is continuous on $[a, b]$ with the exception of a vertical asymptote at $x = c$, then

$$\int_a^b = \lim_{k \to c^-} \int_a^k f(x)dx + \lim_{k \to c^+} \int_k^b f(x)dx.$$

If the limit exists, then the improper integral is said to converge. Otherwise, the improper integral is said to diverge.

3 Exercises

Evaluate the following improper integrals

I. $\displaystyle\int_1^\infty \frac{dx}{x^2}$

II. $\displaystyle\int_1^\infty \frac{dx}{x}$

III. $\displaystyle\int_0^\pi \tan(x)dx$

IV. $\displaystyle\int_1^\infty \frac{dx}{x-3}$