Calculus with Analytic Geometry II

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1 U-Sub Review

Recall that u-substitution is designed to perform integration of the following form:

$$\begin{aligned} \int f'(g(x))g'(x)dx &= \int f'(u)du \quad (u = g(x),\ du = g'(x)dx) \\ &= f(u) + C \quad \left(\frac{d}{dx} f(u) = f'(u)\right) \\ &= f(g(x)) + C \quad (u = g(x)) \end{aligned}$$

Hence, u-substitution is the antiderivative analog of the chain rule.

2 Integration by Parts

Similarly, integration by parts is the antiderivative analog of the product rule. To this end, consider the following integral

$$\int (f(x) \cdot g(x))' \, dx = \int (f'(x) \cdot g(x) + f(x) \cdot g'(x)) \, dx = \int f'(x) \cdot g(x)dx + \int f(x) \cdot g'(x)dx.$$

Note that $\int (f(x) \cdot g(x))' \, dx = f(x) \cdot g(x)$. Hence, we arrive at the following integration by parts formula

$$\int f(x) \cdot g'(x)dx = f(x)g(x) - \int f'(x) \cdot g(x)dx.$$

We can write this formula in a more compact notation by using the following substitutions: Let $u = f(x)$ and $v = g(x)$, then the integration by parts formula becomes

$$\int udv = uv - \int vdu.$$

For integration by parts to be useful, you must be able to differentiate $u = f(x)$ and find an antiderivative of $dv = g'(x)dx$. Furthermore, the resulting integral $\int vdu$ should be easier (usually) to integrate than the original integral $\int udv$. As an example, consider the integral

$$\int x \cos(x)dx.$$

If we identify $u = \cos(x)$ and $dv = x$, then integration by parts implies

$$\int x \cos(x)dx = \frac{1}{2}x^2 \cos(x) + \frac{1}{2} \int x^2 \sin(x)dx.$$

While the above formula is correct, it is not helpful since the resulting integral is harder than the original integral. Instead, we identify $u = x$ and $dv = \cos(x)$. Then, integration by parts implies

$$\int x \cos(x)dx = x \sin(x) - \int \sin(x)dx = x \sin(x) + \cos(x) + C.$$

3 Exercises

Evaluate the following using integration by parts.

I. $\int xe^x dx$

II. $\int \ln(x) dx$

III. $\int x^2 e^{-x} dx$

IV. $\int e^x \cos(x) dx$