Calculus with Analytic Geometry II

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1 Integration by Parts

Integration by parts is the antiderivative analog of the product rule. To this end, consider the following integral

$$\int (f(x) \cdot g(x))' \, dx = \int (f'(x) \cdot g(x) + f(x) \cdot g'(x)) \, dx = \int f'(x) \cdot g(x) dx + \int f(x) \cdot g'(x) dx.$$

Note that $\int (f(x) \cdot g(x))' \, dx = f(x) \cdot g(x)$. Hence, we arrive at the following integration by parts formula

$$\int f(x) \cdot g'(x) dx = f(x) g(x) - \int f'(x) \cdot g(x) dx.$$

We can write this formula in a more compact notation by using the following substitutions: Let $u = f(x)$ and $v = g(x)$, then the integration by parts formula becomes

$$\int u dv = uv - \int v du.$$

For integration by parts to be useful, you must be able to differentiate $u = f(x)$ and find an antiderivative of $dv = g'(x)dx$. Furthermore, the resulting integral $\int v du$ should be easier (usually) to integrate than the original integral $\int u dv$. As an example, consider the integral

$$\int x \cos(x) dx.$$

If we identify $u = \cos(x)$ and $dv = x$, then integration by parts implies

$$\int x \cos(x) dx = \frac{1}{2} x^2 \cos(x) + \frac{1}{2} \int x^2 \sin(x) dx.$$

While the above formula is correct, it is not helpful since the resulting integral is harder than the original integral. Instead, we identify $u = x$ and $dv = \cos(x)$. Then, integration by parts implies

$$\int x \cos(x) dx = x \sin(x) - \int \sin(x) dx = x \sin(x) + \cos(x) + C.$$

2 Inverse Trig Formulas

Recall the following derivative rules:

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - x^2}}, \ \frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1 - x^2}}, \ \frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}.$$

Using integration by parts, we can use these derivative rules to find antiderivative rules for each inverse trig function. For example,

$$\begin{aligned}\int \arcsin(x)dx &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx \qquad (u=\arcsin(x),\ dv=dx)\\ &= x\arcsin(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad \left(u = 1-x^2\right)\\ &= x\arcsin(x) + \sqrt{1-x^2} + C.\end{aligned}$$

3 Reduction Formulas

We can use integration by parts to produce reduction formulas for the integral of powers of sine or cosine. For instance, for $n \geq 2$, we have

$$\begin{aligned}\int \sin^n(x)dx &= \int \sin^{n-1}(x)\sin(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)dx - (n-1)\int \sin^n(x)dx\end{aligned}$$

Therefore, we have

$$\int \sin^n(x)dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx.$$

We can also derive reduction formulas for powers of tangent and secant. For instance, for $n \geq 2$, we have

$$\begin{aligned}\int \sec^n(x)dx &= \int \sec^{n-2}(x)\sec^2(x)dx\\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)dx\\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)dx\\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx\end{aligned}$$

Therefore, we have

$$\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx.$$