Calculus with Analytic Geometry II

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1 Antiderivatives

Let $f(x)$ be a function. Then, $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$. Note that, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. The indefinite integral of $f(x)$, denoted $\int f(x)dx$, represents the family of antiderivatives of $f(x)$. In particular,

$$\int f(x)dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant. Note that the differential dx determines that $F'(x) = \frac{d}{dx}F(x)$, that is, we are differentiating $F(x)$ with respect to x.

We can derive many basic integration rules simply from our knowledge of derivatives. Below is a summary of several integration rules:

$$\begin{aligned}
\frac{d}{dx}C = 0 \;&\Rightarrow\; \int 0dx = C \\
\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x) \;&\Rightarrow\; \int \left(f(x) + g(x)\right)dx = \int f(x)dx + \int g(x)dx \\
\frac{d}{dx}x^n = nx^{n-1} \;&\Rightarrow\; \int x^n dx = \frac{x^{n+1}}{n+1} + C \\
\frac{d}{dx}\sin(x) = \cos(x) \;&\Rightarrow\; \int \cos(x)dx = \sin(x) + C \\
\frac{d}{dx}e^x = e^x \;&\Rightarrow\; \int e^x = e^x + C \\
\frac{d}{dx}\ln(x) = \frac{1}{x} \;&\Rightarrow\; \int \frac{1}{x}dx = \ln|x| + C
\end{aligned}$$

2 Area Bounded by Curves

Let $f(x)$ be a continuous function on the interval $[a, b]$. The definite integral of $f(x)$ over $[a, b]$, denoted $\int_a^b f(x)dx$, is the signed area of the region bounded by the curve $f(x)$ and the x-axis over the interval $[a, b]$. By signed area, we mean that any portion of the region above the x-axis has positive area, while any portion below the x-axis has negative area.

For simple regions, we can evaluate the definite integral using geometric formulas for area. For example,

$$\int_{0,2} (2 - x)dx$$

represents the area of the triangle bounded by the x-axis, y-axis, and the line $y = 2 - x$. The area of this triangle is 2; hence, $\int_0^2 (2 - x)dx = 2$.

For more complicated regions, we use the limit definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x,$$

where the interval $[a, b]$ is partitioned into n subintervals $[x_{i-1}, x_i]$, $1 \le i \le n$, the length of each subinterval is Δx, and c_i is any point in the subinterval $[x_{i-1}, x_i]$. As an example, consider the definite integral $\int_0^1 x^2 dx$. We partition the itnerval $[0, 1]$ into n subitnervals $[x_{i-1}, x_i] = [\frac{i-1}{n}, \frac{i}{n}]$, and we select $c_i = \frac{i}{n}$ for each $1 \le i \le n$. Hence,

$$\begin{aligned}\int_0^1 x^2 dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2\\ &= \lim_{n\to\infty}\frac{1}{n^3}\left(\frac{1}{6}n(n+1)(2n+1)\right) = \frac{1}{3}.\end{aligned}$$

3 Fundamental Theorem of Calculus

One of the most impportant theorems in all of calculus describes the relationship between the definite and indefinite integrals. In particular, if $f(x)$ is a continuous function on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Hence, the complex problem of determining the area of any region can be solved by finding an antiderivative of a function. For Example,

$$\int_0^{\pi/2}\cos(x)dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1.$$

4 U-Substitution

The one method for integration covered in Calc I is U-Substitution. In this method, we seek a change of variables to transform a complex integral to a simple integral. U-substitution is particularly useful when the given integration problem can be viewed in the following form:

$$\int f(g(x))g'(x)dx.$$

Then, we make the substitution $u = g(x)$ so that $du = g'(x)dx$ and our integral becomes

$$\int f(u)du.$$

For example,

$$\begin{aligned}\int \cos(x)e^{\sin(x)}dx &= \int e^u du, \qquad u = \sin(x)\\ &= e^u + C = e^{\sin(x)} + C.\end{aligned}$$

Some substitutions are less apparent. For example, consider the following integral:

$$\int x^2\sqrt{x-1}dx.$$

We let $u = x - 1$, so that $du = dx$ and $x^2 = (u+1)^2$. Then, we have

$$\begin{aligned}
\int x^2\sqrt{x-1}dx &= \int (u+1)^2\sqrt{u}du \\
&= \int (u^2+2u+1)u^{1/2}du \\
&= \int u^{5/2} + 2u^{3/2} + u^{1/2}du \\
&= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.
\end{aligned}$$

Finally, consider the following integral:

$$\int \frac{1}{4+x^2}dx = \frac{1}{4}\int \frac{1}{1+(x/2)^2}dx.$$

We let $u = x/2$, so that $du = dx/2$. Then, we have

$$\begin{aligned}
\frac{1}{4}\int \frac{1}{1+(x/2)^2}dx &= \frac{1}{4}\int \frac{1}{1+u^2}2du \\
&= \frac{1}{2}\int \frac{1}{1+u^2}du \\
&= \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x/2) + C.
\end{aligned}$$