

# Calculus with Analytic Geometry II

Thomas R. Cameron

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## 1 The Integral Test

The integral test is a comparison test between the area under a curve and the value of an infinite series. In particular, let  $\sum_{k=1}^{\infty} u_k$  be a series with positive terms. If  $f$  is a function that is decreasing and continuous on  $[1, \infty)$  and  $u_k = f(k)$  for all  $k \geq 1$ , then

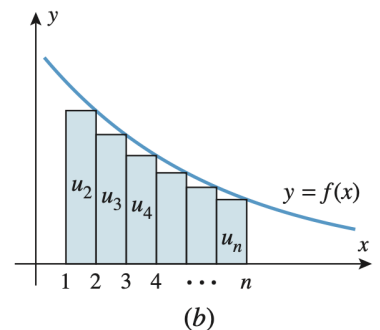
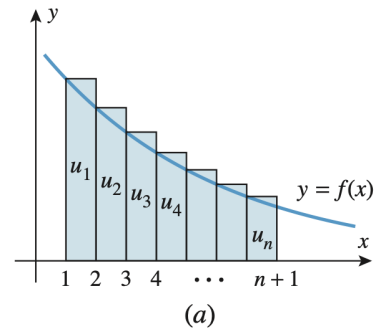
$$\sum_{k=1}^{\infty} u_k \text{ and } \int_a^{\infty} f(x) dx$$

either both converge or both diverge.

To see why the integral test is true, consider the figure on the right. Since  $f(k) = u_k$  for all  $k \geq 1$ , the values of  $u_k$  can be viewed as the area of a rectangle of height  $f(k)$  and width 1. Let  $s_n$  denote the  $n$ th partial sum, then

$$\begin{aligned} s_n - u_1 &= u_2 + u_3 + \cdots + u_n < \int_1^n f(x) dx \\ &< \int_1^{n+1} f(x) dx < u_1 + u_2 + \cdots + u_n = s_n. \end{aligned}$$

Therefore, the series and integral either both converge or both diverge.



Now, we can revisit the harmonic series  $\sum_{k=1}^{\infty} \frac{1}{k}$ . Note that  $f(x) = 1/x$  can be used to apply the integral test. Furthermore,

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{n \rightarrow \infty} \ln(n) = \infty.$$

Hence, the harmonic series diverges.

In general, we can apply the integral test to the p-series  $\sum_{k=1}^{\infty} \frac{1}{k^p}$ , where  $p > 1$ . Note that

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{n \rightarrow \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^n$$