Calculus with Analytic Geometry II

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1 The Integral Test

The integral tests is a comparison test between the area under a curve and the value of an infinite series. In particular, let $\sum_{k=1}^{\infty} u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on $[1, \infty)$ and $u_k = f(k)$ for all $k \geq 1$, then

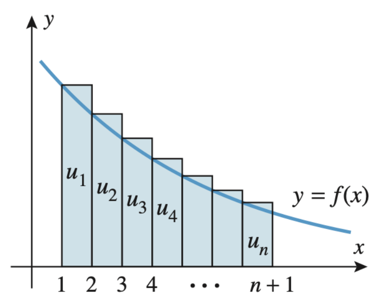
$$\sum_{k=1}^{\infty} u_k \text{ and } \int_a^{\infty} f(k)$$

either both converge or both diverge.

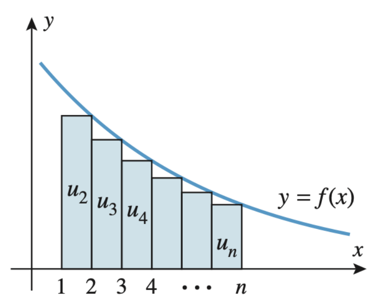
To see why the integral test is true, consider the figure on the right. Since $f(k) = u_k$ for all $k \geq 1$, the values of u_k can be viewed as the area of a rectange of height $f(k)$ and width 1. Let s_n denote the nth partial sum, then

$$\begin{aligned} s_n - u_1 = u_2 + u_3 + \cdots + u_n &< \int_1^n f(x)dx \\ &< \int_1^{n+1} f(x)dx < u_1 + u_2 + \cdots + u_n = s_n. \end{aligned}$$

Therefore, the series and integral either both converge or both diverge.



(a)



(b)

Now, we can revisit the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Note that $f(x) = 1/x$ can be used to apply the integral test. Furthermore,

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{n \to \infty} \ln(n) = \infty.$$

Hence, the harmonic series diverges.

In general, we can apply the integral test to the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where $p > 1$. Note that

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{n \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^n,$$

which converges provided that $p > 1$.