MATH-141 **Name:** ______________________
Spring 2025
Intro Exam Solution
January 13 **Pledge:** ______________________

Each question topic and point value is recorded in the tables below. Note that this exam must be completed within the 1 hour and 50 minutes allotted. Also, you must work without any external resources, which includes no notes, calculator, nor any equivalent software. You must show an appropriate amount of work to justify your answer for each problem. If you run out of room for a given problem, you may continue your work on the back of the page. By writing your name and signing the pledge you are stating that you understand the rules outlining this exam.

Scoring Table

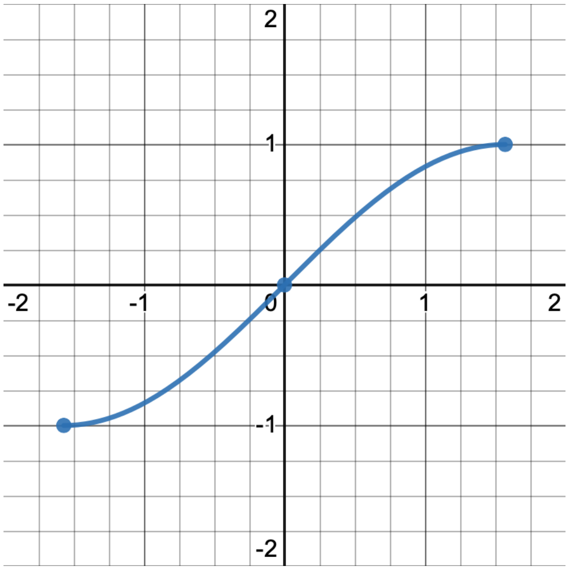
Question	Points	Score
1	10	
2	6	
3	9	
4	9	
Total:	34	

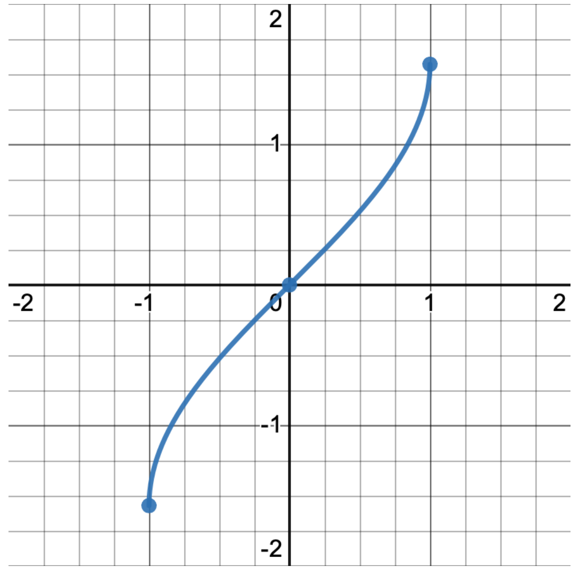
Topics Table

Question	Topic
1	Inverse Functions, Logs, and Exponentials
2	Limit Definition of the Derivative
3	Differentiation
4	U-Substitution

1. (a) (5 points) Sketch the graph of $\sin(x)$ on a restricted domain so the function is one-to-one. Then, sketch a graph of $\arcsin(x)$.

Solution:





(b) (3 points) Solve the following logarithmic equation for x:

$$2\ln(x) = \ln\left(x - \frac{2}{3}\right) + \ln(3).$$

Solution: Using the (log) power and product rules, we have

$$\ln(x^2) = \ln(3x - 2).$$

Therefore, $x^2 = 3x - 2$, which is equivalent to $x^2 - 3x + 2 = 0$, i.e., $(x - 1)(x - 2) = 0$. Hence, $x = 1$ and $x = 2$ are the solutions.

(c) (2 points) Solve the following exponential equation for x:

$$\frac{3^{2x+1}}{3^{x-1}} = 9.$$

Solution: Using the (exponent) quotient rule, we have

$$3^{x+2} = 3^2,$$

which implies that $x + 2 = 2$, i.e., $x = 0$.

2. (6 points) Use the limit definition to find the derivative of $f(x) = \sqrt{x+5}$.

Solution: By definition

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+5+h} - \sqrt{x+5}}{h} \end{aligned}$$

Using the conjugate, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+5+h} - \sqrt{x+5}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+5+h} - \sqrt{x+5}}{h} \cdot \frac{\sqrt{x+5+h} + \sqrt{x+5}}{\sqrt{x+5+h} + \sqrt{x+5}} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+5+h} + \sqrt{x+5}\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+5+h} + \sqrt{x+5}} = \frac{1}{2\sqrt{x+5}}. \end{aligned}$$

3. (a) (2 points) Evaluate the derivative of $f(x) = xe^x$

Solution: Using the product rule, we have

$$f'(x) = 1e^x + xe^x = e^x(1 + x).$$

(b) (3 points) Evaluate the derivative of $f(x) = \ln\left(\frac{x-1}{x+1}\right)$

Solution: By the (log) quotient rule,

$$\ln\left(\frac{x-1}{x+1}\right) = \ln(x-1) - \ln(x+1).$$

Hence, the chain rule gives

$$f'(x) = \frac{1}{x-1} - \frac{1}{x+1}$$

(c) (4 points) Find the derivative of $\arccos(x)$ using implicit differentiation.

Solution: Let $y = \arccos(x)$. Then. $\cos(y) = x$. Differentiating both sides with respect to x gives us

$$-\sin(y)\frac{dy}{dx} = 1.$$

Using a right triangle with angle y, we see that $\cos(y) = x$ implies that $\sin(y) = \sqrt{1 - x^2}$. Therefore, we have

$$\frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1 - x^2}}.$$

4. Evaluate the following definite and indefinite integrals

(a) (3 points) $\int_0^1 3x\sqrt{x^2+1}dx$

Solution: Let $u = x^2 + 1$. Then, $du = 2xdx$ and $\frac{3}{2}du = 3xdx$. Hence, we have

$$\begin{aligned}\int_0^1 3x\sqrt{x^2+1}dx &= \frac{3}{2}\int_1^2 \sqrt{u}du \\ &= \frac{3}{2}\frac{2}{3}u^{3/2}\Big|_1^2 \\ &= 2^{3/2} - 1^{3/2}.\end{aligned}$$

(b) (3 points) $\int \frac{\cos(x)}{\sin(x)}dx$

Solution: Let $u = \sin(x)$. Then, $du = \cos(x)dx$. Hence, we have

$$\begin{aligned}\int \frac{\cos(x)}{\sin(x)}dx &= \int \frac{1}{u}du \\ &= \ln|u| + C = \ln|\sin(x)| + C.\end{aligned}$$

(c) (3 points) $\int \frac{e^x dx}{\sqrt{e^{2x}+1}}$

Solution: Let $u = e^x$. Then, $du = e^x dx$. Hence, we have

$$\begin{aligned}\int \frac{e^x dx}{\sqrt{e^{2x}+1}} &= \int \frac{1}{\sqrt{u^2+1}}du \\ &= \operatorname{arcsinh}(u) + C = \operatorname{arcsinh}(e^x) + C.\end{aligned}$$