Calculus with Analytic Geometry II

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February 14, 2025

1 Convergence of Sequences

When thinking about the convergence of a sequence it is helpful to view a plot of the sequence over several positive integers. For example, consider the sequences shown in Figure 1.

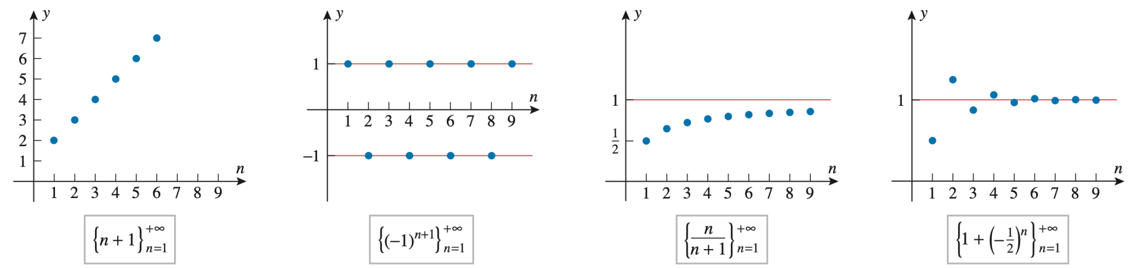

Figure 1: The plot of several sequences.

Note that the terms in the sequence $\{\frac{n}{n+1}\}$ appear to get closer to 1 as $n \to \infty$. In contrast, the terms in the sequence $\{(-1)^{n+1}\}$ don't appear to be getting closer to any single value. These observations motivate the definition of convergence for a sequence. In particular, the sequence $\{a_n\}$ is said to converge to L, which we denote by $\lim_{n\to\infty} a_n = L$, if for all $\epsilon > 0$ there is an N such that $|a_n - L| < \epsilon$ whenever $n > N$. Intuitively, ϵ should be viewed as a bound and N should be viewed as a marker. Then, convergence to L means that all terms past the marker N are within ϵ of L, see Figure 2.

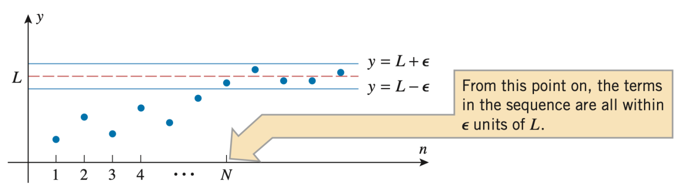

Figure 2: Visualization of convergence definition.

From the visualization in Figure 2, it should be easy to argue why the sequence $\{(-1)^{n+1}\}_{n=1}^{\infty}$ does not converge. Indeed, let $\epsilon = 1$. Then, there is no N such that all terms past N have a value within ϵ of any single value L. In fact, we can see that in order for convergence to occur the odd and even indexed terms of a sequence must converge to the same value.

To prove convergence, we must be able to identify a marker N for each tolerance ϵ. For example, consider the sequence $\{\frac{n}{n+1}\}$. We want to find a marker N such that

$$\left|\frac{n}{n+1} - 1\right| < \epsilon$$

for all $n > N$. To this end, note that $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1}$. Furthermore, $\frac{1}{n+1} < \epsilon$ whenever $n + 1 > \frac{1}{\epsilon}$. Hence, we have identified a marker $N = \frac{1}{\epsilon} - 1$.

Now, a formal proof of $\lim_{n\to\infty} \frac{n}{n+1} = 1$ can be written as follows:

Proof. Let $\epsilon > 0$ and define $N = \frac{1}{\epsilon} - 1$. Then, $n > N$ implies that $n + 1 > \frac{1}{\epsilon}$ Therefore, we have

$$\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \epsilon.$$

$\square$

Complex sequences require more advanced techniques to prove or disprove convergence. For example, Let $f(n) = a_n$ denote a sequence and let $f(x)$ denote the extended function over the real numbers. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. Hence, we can apply L'Hoptials rule to determine the convergence or divergence of sequences. Also, we can use the squeeze theorem to determine the convergence of sequences. In particular, suppose that $a_n \leq b_n \leq c_n$ for all $n > N$, where N is some marker. If a_n and c_n both converge to L, then b_n must also converge to L.

2 Exercises

I. Use the convergence definition to show that

$$\lim_{n\to\infty} \left(1 + \left(\frac{-1}{2}\right)^n\right) = 1.$$

II. Use L'Hoptials rule to show that

$$\lim_{n\to\infty} \frac{n}{e^n} = 0.$$

III. Use the squeeze theorem to show that

$$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0.$$