Calculus with Analytic Geometry II

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April 21, 2025

1 Multivariable Functions

A multivariable function is a function of two or more variables. We will be focused on functions of two variables, which map points (x, y) in the plane to a unique real number $z = f(x, y)$. Note that x and y are called the independent variables and z is called the dependent variable.

For example, consider the function

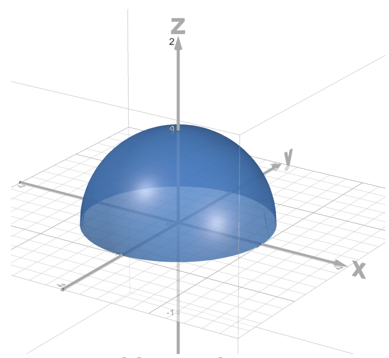
$$f(x, y) = \sqrt{1 - x^2 - y^2},$$

which is shown on the right. Note that the domain of this function is

$$x^2 + y^2 \leq 1,$$

i.e., all points in the plane that lie on or inside the unit circle. Moreover, the range of this function is

$$0 \leq z \leq 1.$$



We can sketch the graph of a multivariable function $z = f(x, y)$ by setting a variable equal to a constant within its allowable range and solving for the other two variables. For example, if we set $x = 0$, then the function above can be written as $z = \sqrt{1 - y^2}$, which is the equation of a semicircle of radius 1 in the yz-plane. Similarly, setting $y = 0$ gives $z = \sqrt{1 - x^2}$, which is a semicircle of radius 1 in the xz-plane. Finally, setting $z = 0$ gives $x^2 + y^2 = 1$, which is the unit circle in the xy-plane. These equations are called traces and they correspond to the intersection of the solid $z = f(x, y)$ with a plane in $\mathbb{R}^3$.

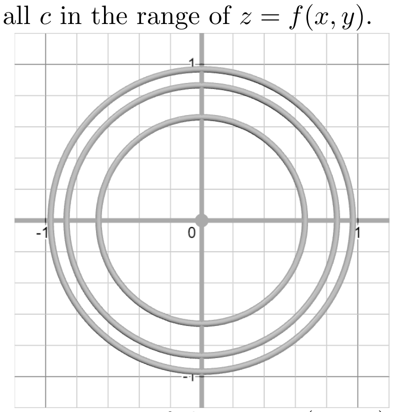
2 Level Curves

The level curves are the traces in the xy-plane corresponding to $z = c$ for all c in the range of $z = f(x, y)$.

For example, the level curves of

$$z = f(x, y) = \sqrt{1 - x^2 - y^2},$$

for $z = 1, 3/4, 1/2, 1/4$ are shown on the right. Note that each level curve $z = c$ corresponds to a circle of radius $\sqrt{1 - c^2}$. In particular, these are the points (x, y) in the domain for which $f(x, y) = c$.



The set of level curves is called a countour map. The contour map is the projection of the points (x, y, c) from the surface $f(x, y) = c$ onto the xy-plane.

3 Limits

If the value of $z = f(x, y)$ approaches a single value L as (x, y) approaches (x_0, y_0), then we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

In particular, this means that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

For example,

$$\lim_{(x,y)\to(0,0)} \sqrt{1 - x^2 - y^2} = 1.$$

Let $\epsilon > 0$ and $\delta < \min\{1, \epsilon\}$. Then, $0 < \sqrt{x^2 + y^2} < \delta$ implies that

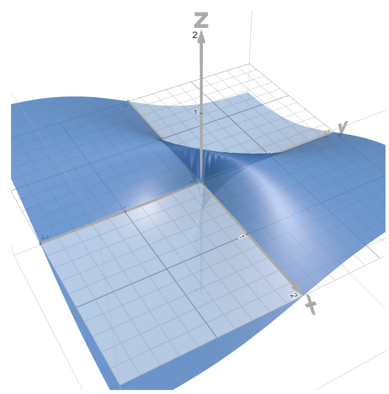
$$\begin{aligned} \left|\sqrt{1 - x^2 - y^2} - 1\right| &= 1 - \sqrt{1 - (x^2 + y^2)} \\ &< 1 - \sqrt{1 - \delta^2} \\ &< 1 - (1 - \delta^2) = \delta^2 < \delta < \epsilon. \end{aligned}$$

Determining if a limit exists is more complicated for multivariable functions since all directions must be considered.

For example, the function $z = f(x, y) = \frac{xy}{x^2+y^2}$ has no limit at the origin, see figure on the right. Note that if we approach the origin along the x-axis or y-axis, then the limiting value of $f(x, y)$ is 0. However, if we approach along the origin along the line $y = x$, then the limiting value of $f(x, y)$ is $1/2$. Therefore, if $\epsilon = 1/4$, there is no circle centered at the origin for which all points (x, y) inside the circle satisfy

$$|f(x, y) - L| < \epsilon,$$

for a single value L.



4 Exercises

Sketch a plot of each function below and its level curves for values $c = 0, 1/2, 1$. Then, determine if the limit as $(x, y) \to (0, 0)$ exists.

I. $z = f(x, y) = 1 - x - y$

II. $z = \frac{x^2 y}{x^4 + y^2}$

III. $z = xy \sin\left(\frac{1}{xy}\right)$