Multivariable Functions

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Exercises

I. Use the traces $x = 0$, $y = 0$, and $z = 0$ to sketch the graph of

$$z = 1 - x - y.$$

Draw the contour map of $f(x, y) = 1 - x - y$ with level curves $c = 0, 1/2, 1, 3/2, 2$. Use the contour map to explain why the limit of $f(x, y)$ at $(0, 0)$ exists.

II. Use the traces $x = 0$, $y = 0$, and $z = 1$ to sketch the graph of

$$z = x^2 + y^2.$$

Then, use the ϵ-δ definition to show that the limit at $(1, 1)$ exists.

III. Consider the function

$$f(x, y) = \frac{x^2 y}{x^4 + y^2}.$$

(a) Show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

along any line $y = mx$.

(b) Show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{1}{2}$$

along the parabola $y = x^2$.

(c) Does the limit of $f(x, y)$ exist at $(0, 0)$?

IV. Consider the function

$$f(x, y) = \begin{cases} xy \sin\left(\frac{1}{xy}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that $z = f(x, y)$ is continuous at the origin.

V. Consider the function

$$f(x, y) = x^2 - y^2.$$

Use the limit definition to find the partial derivatives of $f(x, y)$ at the point $(1, 0)$.

Figures from Lecture

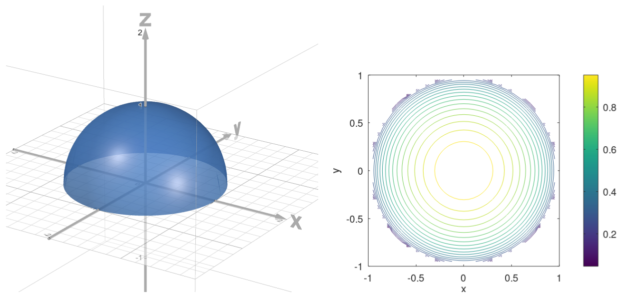

Figure 1: The graph and level curves of $z = \sqrt{1 - x^2 - y^2}$, where $x^2 + y^2 \leq 1$.

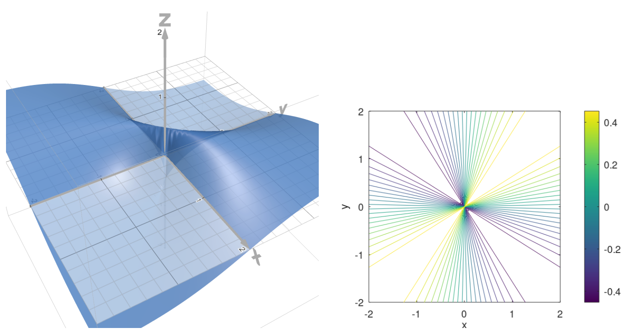

Figure 2: The graph and level curves of $z = \frac{xy}{x^2+y^2}$, where $-2 \leq x, y \leq 2$.

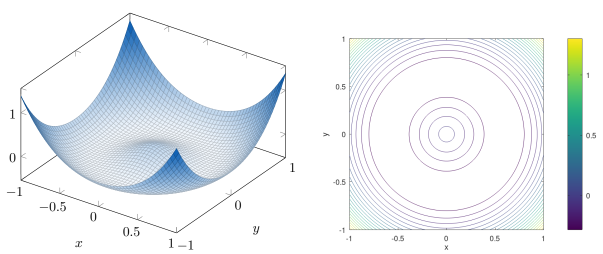

Figure 3: The graph and level curves of $z = (x^2 + y^2)\ln(x^2 + y^2)$, for $(x, y) \neq (0, 0)$.

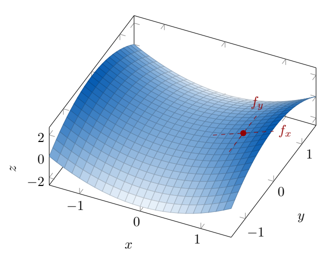

Figure 4: Tangent lines to $z = x^2 - y^2$ at $(1, 0)$.