Calculus with Analytic Geometry II

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April 7, 2025

1 Review

Recall that parametric equations can be used to parameterize curves in the plane. For example, the equations $x = -1 + 2\cosh(t)$ and $y = 1 + \sinh(t)$, where $-\infty < t < \infty$, parameterize the right half of the hyperbola centered at $(-1, 1)$ with a horizontal focal axis with length 4.

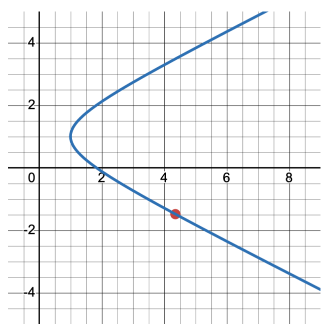

Figure 1: Hyperbola parameterized by $x = -1 + 2\cosh(t)$ and $y = 1 + \sinh(t)$.

It is worth noting that this hyperbola can also be parameterized by $x = -1 + 2\sec(t)$ and $y = 1 + \tan(t)$, where $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2 Calculus on Parametric Equations

Consider the parameterization $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where both $f(t)$ and $g(t)$ are differentiable and $f'(t) \neq 0$. For each t between a and b, the change in x with respect to t is given by $\frac{dx}{dt} = f'(t)$, and the change in y with respect to t is given by $\frac{dy}{dt} = g'(t)$. Therefore, since $f'(t) \neq 0$, the change in y with respect to x is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}.$$

As an example, consider the parameterization of the hyperbola given by $x = -1+2\sec(t)$ and $y = 1+\tan(t)$, where $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Then,

$$\frac{dx}{dt} = 2\sec(t)\tan(t), \qquad \frac{dy}{dt} = \sec^2(t), \qquad \frac{dy}{dx} = \frac{1}{2}\frac{\sec(t)}{\tan(t)} = \frac{1}{2\sin(t)}.$$

It is worth noting that value of t between $-\pi/2$ and $\pi/2$ corresponds to a point on the hyperbola given by the parametric equations. Therefore, the value of $\frac{dy}{dx}$ at a given value of t can be viewed as the slope of the tangent line to the curve at that point. For instance, when $t = 0$, we are at the point $(1, 1)$ and at this point the tangent line is vertical. At $t = \pi/4$, we are at the point $(-1 + \frac{4}{\sqrt{2}}, 2)$ and at this point the tangent line has slope $\frac{1}{\sqrt{2}}$.

Since $\frac{dy}{dx}$ is a function of t, we can also determine the 2nd derivative at a point on a parameterized curve as follows

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}.$$

For example, the second derivative at a point on the hyperbola is given by

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{1}{2}\sin^{-1}(t)}{\frac{d}{dt}(1+2\sec(t))} = -\frac{1}{4}\frac{\cos^3(t)}{\sin^3(t)}.$$

3 Exercises

1. Sketch the curve parameterized by $x = 2t - \pi\sin(t)$ and $y = 2 - \pi\cos(t)$, for $-\pi \leq t \leq \pi$.
2. Determine the point at which this curve crosses itself.
3. Find the tangent line equations at this point.