# Calculus with Analytic Geometry II

Thomas R. Cameron

April 4, 2025

## 1 Review

Recall that conic sections such as circles, ellipses, parabolas, and hyperbolas, correspond mathematical equations derived from their geometric definition. To summarize, the parabola with center $(h, k)$ and directrix $y = k - p$ is defined by

$$(x - h)^2 = 4p(y - k).$$

If the directrix is $x = h - p$, then the parabola is defined by

$$(y - k)^2 = 4p(x - h).$$

The ellipse with center $(h, k)$, semimajor axis length $a$, and semiminor axis length $b$, is defined by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1,$$

if the major axis is horizontal. If the major axis is vertical, then the ellipse is defined by

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1.$$

Moreover, the foci of the ellipse lie on the major axis a distance of $c$ away from the center, where $c = \sqrt{a^2 - b^2}$.

The hyperbola with center $(h, k)$, focal (transverse) axis length $a$, and conjugate axis length $b$, is defined by

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1,$$

if the focal axis is horizontal. If the focal axis is vertical, then the hyperbola is defined by

$$\frac{(x - h)^2}{b^2} - \frac{(y - k)^2}{b^2} = 1.$$

Moreover, the foci of the hyperbola lie on the focal axis a distance of $c$ away from the center, where $c = \sqrt{a^2 + b^2}$.

## 2 Parametric Equations

All the conic sections we've described are examples of plane curves (curves in the plane). We can transverse these curves as a function of "time", which is known as a parameterization. In particular, a parameterization of the curve $C$ is give the equations

$$x = f(t), \qquad y = g(t)$$

such that $(x(t), y(t))$ is a point on the curve $C$ for all $t$ in a interval $[a, b]$. Such a pair of equations are called parametric equations.

As an example, consider the paramteric equations

$$x = h + a\cos(t), \qquad y = k + b\sin(t).$$

These parametric equations satisfy the following

$$\begin{aligned}\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= \frac{a^2\cos^2(t)}{a^2} + \frac{b^2\sin^2(t)}{b^2} \\ &= \cos^2(t) + \sin^2(t) = 1.\end{aligned}$$

Therefore, these paramteric equations form a parametrization of the ellipse centered at $(h,k)$, with semimajor axis length $a$, and semiminor axis length $b$.

As another example, consider the parametric equations

$$x = h + a\cosh(t), \qquad y = k + b\sinh(t).$$

These parametric equations satisfy the following

$$\begin{aligned}\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= \frac{a^2\cosh^2(t)}{a^2} - \frac{b^2\sinh^2(t)}{b^2} \\ &= \cosh^2(t) - \sinh^2(t) = 1.\end{aligned}$$

# 3 Exercises

1. Parameterize the parabola given by
$$(x-1)^2 = 8(y-1)$$

2. Parameterize the ellipse given by
$$9x^2 + 4y^2 - 18x + 24y + 9 = 0$$

3. Parameterize the hyperbola given by
$$x^2 - 4y^2 + 2x + 8y - 7 = 0$$