Calculus with Analytic Geometry II

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1 Partial Fraction Decomposition

The method of partial fraction decomposition seeks to decompose a proper rational function (degree of numerator is less than degree of denominator) into a sum of fractions whose denominator are powers of either linear or irreducible quadratics. For example, consider the following fraction:

$$\frac{5x-10}{x^2-3x-4} = \frac{5x-10}{(x-4)(x+1)}.$$

Since the denominator of this fraction is a product of two linear factors, we seek the following decomposition

$$\frac{5x-10}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1},$$

where A and B are constants to be determined. To determine the values of A and B, we multiply both sides of the above equation by $(x-4)(x+1)$:

$$5x-10 = A(x+1) + B(x-4) = x(A+B) + (A-4B).$$

We obtain the following equations to solve for A and B:

$$\begin{aligned} 5 &= A + B \\ -10 &= A - 4B \end{aligned}$$

Subtracting the 2nd equation from the 1st equation gives $15 = 5B$. Hence, $B = 3$ and $A = 2$ and it follows that

$$\frac{5x-10}{(x-4)(x+1)} = \frac{2}{x-4} + \frac{3}{x+1}.$$

2 The Form of a Partial Fraction Decomposition

Consider the proper rational function $P(x)/Q(x)$. The first step of a partial fraction decomposition is to factor $Q(x)$ into irreducible linear and quadratic factors. Then, collect all repeated factors so that $Q(x)$ is expressed as a product of distinct factors of the form

$$(ax+b)^m \text{ and } (ax^2+bx+c)^m.$$

For each factor of $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}.$$

For each factor of $(ax^2+bx+c)^m$, the partial fraction dcomposition constains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

For example, consider the rational function

$$\begin{aligned}\frac{2x+4}{x^3-2x^2} &= \frac{2x+4}{x^2(x-2)} \\ &= \frac{2x+4}{x \cdot x \cdot (x-2)} \\ &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-2}.\end{aligned}$$

Multiplying both sides of the above equation by $x^2(x-2)$ gives

$$2x+4 = A_1 x(x-2) + A_2(x-2) + A_3 x^2 = x^2(A_1+A_3) + x(A_2 - 2A_1) - 2A_2.$$

Pluggin in $x = 2$ reveals $4A_3 = 8$, i.e., $A_3 = 2$. Pluggin in $x = 0$ reveals $-2A_2 = 4$, i.e., $A_2 = -2$. There is no value of x that isolates A_1; however, $A_1 + A_3 = 0$ since there is no x^2 term on the left hand side. Therefore, $A_1 = -A_3 = -2$, so we have the following decomposition

$$\frac{2x+4}{x^3-2x^2} = -\frac{2}{x} - \frac{2}{x^2} + \frac{2}{x-2}.$$

As a final example, consder the rational function

$$\begin{aligned}\frac{x^2+x-2}{3x^3-x^2+3x-1} &= \frac{x^2+x-2}{x^2(3x-1)+(3x-1)} \\ &= \frac{x^2+x-2}{(3x-1)(x^2+1)} \\ &= \frac{A_1}{3x-1} + \frac{A_2 x + B_2}{x^2+1}\end{aligned}$$

Multiplying both sides of the above equation by $(3x-1)(x^2+1)$ gives

$$x^2+x-2 = A_1(x^2+1) + (A_2 x + B_2)(3x-1) = x^2(A_1 + 3A_2) + x(3B_2 - A_2) + (A_1 - B_2).$$

By equating corresponding coefficients, we end up with the following system of equations

$$\begin{aligned} A_1 + 3A_2 + 0B_2 &= 1 \\ 0A_1 - A_2 + 3B_2 &= 1 \\ A_1 + 0A_2 - B_2 &= -2 \end{aligned}$$

Solving this system of equations yields

$$A_1 = -\frac{7}{5},\ A_2 = \frac{4}{5},\ A_3 = \frac{3}{5}.$$

Therefore,

$$\frac{x^2+x-2}{3x^3-x^2+3x-1} = -\frac{1}{5} \cdot \frac{7}{3x-1} + \frac{1}{5} \cdot \frac{4x+3}{x^2+1}.$$

3 Exercises

I. Evaluate the integral $\displaystyle\int \frac{5x-10}{x^2-3x-4}dx$.

II. Evaluate the integral $\displaystyle\int \frac{2x+4}{x^3-2x^2}dx$.

III. Evaluate the integral $\displaystyle\int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx$.