Calculus with Analytic Geometry II

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June 25, 2025

1 Polar Coordinates

In polar coordinates, every point in the plane is identified by a radial distance r and angle θ measured counter-clockwise from the positive x-axis. For example, the polar coordinates $(r, \theta) = (2, \pi/4)$ identifies the point on the circle of radius 2 at $\pi/4$ radians from the positive x-axis. In cartesian coordinates, this point is $x = 2\cos(\pi/4) = \sqrt{2}$ and $y = 2\sin(\pi/4) = \sqrt{2}$.

In general, the polar coordinate (r, θ) corresponds to the cartesian coordinate $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover, given the cartesian coordinates (x, y) we can identify the polar coordiantes by noting that

$$r^2 = x^2 + y^2 \text{ and } \tan(\theta) = \frac{y}{x}.$$

Note that arctan only returns a radian between $-\pi/2$ and $\pi/2$, so we must consider the sign of x and y to determine the correct angle θ.

We can also graph equations in polar coordinates r and θ. For example, consider the graphs shown in Figure 1

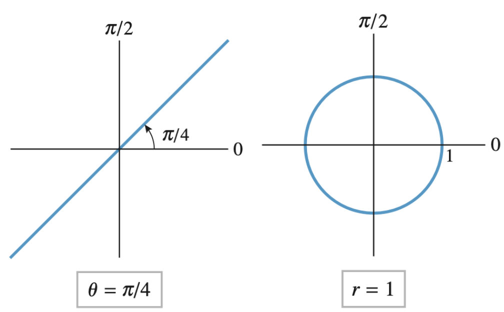

Figure 1: Graphs of basic equations in polar coordinates.

To graph more complicated equations, it is helpful to consider the value of r at varying values of θ For example, consider the graphs shown in Figure 2

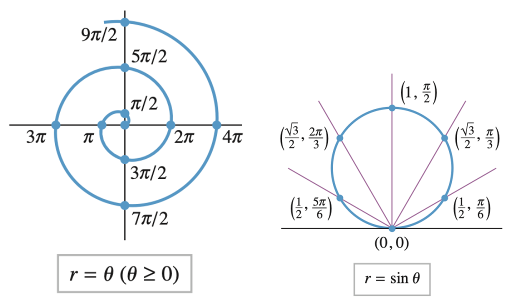

Figure 2: Graphs of more complicated equations in polar coordinates.

Finally, consider the graph of the polar equation $r = \cos(2\theta)$ shown in Figure 3.

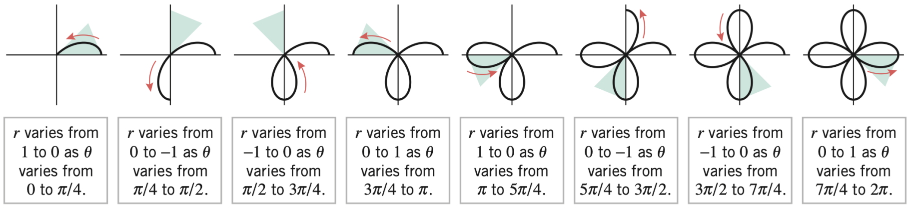

Figure 3: Construction of the graph of $r = \cos(2\theta)$.

2 Tangent Lines in Polar Coordinates

Consider the function $r = f(\theta)$, where f is a differentiable function in θ. In cartesian coordinates, we have $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$. Therefore,

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

For example, if $r = \sin(\theta)$, then

$$\frac{dy}{dx} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}$$

Hence, the graph of $r = \sin(\theta)$ has a horizontal tangent line when $\theta = 0$ and $\theta = \pi$, and a vertical tangent line when $\theta = \pi/4$ and $\theta = 5\pi/4$.

3 Area in Polar Coordinates

Consider the function $r = f(\theta)$, where f is continuous and non-negative over the interval $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 4 (left).

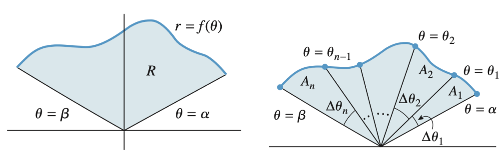

Figure 4: Region bounded by $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$.

To approximate the area of this region, we subdivide the interval $[\alpha, \beta]$ into n subintervals

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \beta.$$

These n subintervals correspond to n sectors of the region as shown in Figure 4 (right). The area of the kth sector can be approximated by a sector of a circle of radius $f(\theta_{k^*})$ and central angle $\Delta\theta_k$, as shown in Figure 5, which is given by $\frac{1}{2}\Delta\theta_k f(\theta_{k^*})^2$.

Therefore, the area of the entire region can be approximated by the following summation

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\Delta\theta_i f(\theta_{i^*})^2.$$

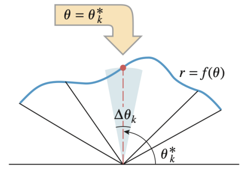

Figure 5: Region bounded by $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$.

Taking the limit as $n \to \infty$ gives us

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \Delta\theta_i f(\theta_{i^*})^2 \\ &= \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta, \end{aligned}$$

where $0 < \beta - \alpha \leq 2\pi$.

For example, consider $r = \cos(2\theta)$. The pedal determined by the interval $[-\pi/4, \pi/4]$ has area

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2(2\theta) d\theta \\ &= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \cos^2(u) du \\ &= \frac{1}{4} \left(\frac{1}{2} \cos(u) \sin(u) + \frac{1}{2} u \right) \Bigg|_{-\pi/2}^{\pi/2} = \frac{\pi}{8}. \end{aligned}$$