Calculus with Analytic Geometry II

Thomas R. Cameron

March 17, 2025

1 Taylor Series

Let $f(x)$ be infinitely differentiable at x_0. Then, the Taylor series of f at x_0 is given by

$$\sum_{k=0}^{\infty} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}$$

For example, the Taylor series of $f(x) = e^x$ at $x_0 = 0$ is given by

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This series is an example of a power series centered at 0.

The Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ is

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{(x-1)^k}{k}.$$

This series is an example of a power series centered at 1.

2 Power Series

Given constants $c_0, c_1, c_2, \ldots$, the series

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k$$

is called a power series centered at x_0. Exactly one of the following is true:

a. The power series only converges when $x = x_0$.

b. The power series converges absolutely for all x.

c. The power series converges absolutely for all x in the interval $(x_0 - R, x_0 + R)$ and diverges for all $|x - x_0| > R$. If $|x - x_0| = R$, then the series may converge or diverge.

The values of x for which the power series converges is known as the interval of convergence.

If the power series is a Taylor series of $f(x)$, then we can use the remainder formula to determine if the series converges to $f(x)$. Recall that the nth Taylor polynomial satisfies

$$f(x) = p_n(x) + \int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt.$$

Hence, the Taylor series converges to $f(x)$ if and only if

$$\lim_{n\to\infty} \int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt = 0.$$

It is often useful to bound the remainder term as follows

$$\left|\int_{x_0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt\right| \le M\frac{|x-x_0|^{n+1}}{(n+1)!},$$

where $\left|f^{(n+1)}(t)\right| \le M$ for all t between x_0 and x.

As an example, consider the Taylor series of $f(x) = e^x$ at $x_0 = 0$ given by

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The ratio test implies that the series converges absolutely for all x. Furthermore, given any x, the remainder is bounded above by

$$\left|\int_{0}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt\right| \le e^x\frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} e^x \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that the Taylor series of $f(x) = e^x$ at $x_0 = 0$ converges to $f(x)$ for all x.

Next, consider the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ given by

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{(x-1)^k}{k}.$$

The ratio test implies that the series converges absolutely for all x in the interval $(0, 2)$. At $x = 0$ the series diverges and at $x = 2$ the series converges conditionally. Furthermore, given any x in $[1, 2]$, the remainder is bounded above by

$$\left|\int_{1}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt\right| \le \frac{n!}{1^{n+1}}\frac{|x-1|^{n+1}}{(n+1)!} = \frac{|x-1|^{n+1}}{n+1},$$

which goes to zero as $n \to \infty$. Also, for any x in $(0, 1)$, the remainder is bounded above by

$$\left|\int_{1}^{x} f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt\right| \le \frac{n!}{x^{n+1}}\frac{|x-1|^{n+1}}{(n+1)!} = \frac{|x-1|^{n+1}}{x^{n+1}(n+1)},$$

which goes to zero as $n \to \infty$. Therefore, the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ converges to $f(x)$ for all x in $(0, 2]$.