Calculus with Analytic Geometry II

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1 Power Series

We have seen that the Taylor series of $f(x)$ at x_0 gives rise to a power series of the formula

$$\sum_{k=0}^{\infty} c_k (x - x_0)^k,$$

where $c_k = \frac{f^{(k)}(x_0)}{k!}$. We are interested in the interval of convergence, which are the values of x for which the power series converges. Moreover, given a x in the interval of convergence, we want to know if the series converges to $f(x)$. Today, we will derive the power series representation of several types of functions and we will investigate their convergence.

2 Geometric Series

We begin with a class of functions that can be represented by a geometric series. Recall the geometric series

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r},$$

for all $|r| < 1$.

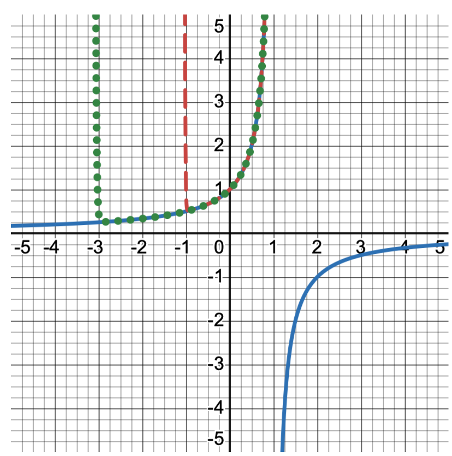
As an example, consider the function $f(x) = \frac{1}{1-x}$, which can be viewed as the result of the geometric series with $a = 1$ and $r = x$. Hence,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

which converges absolutely for all $|x| < 1$, i.e., $-1 < x < 1$. We can represent $f(x)$ in another interval by centering our power series about a different point. For instance, note that

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-(x+1)/2} = \sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^k,$$

which converges for all $\left|\frac{x+1}{2}\right| < 1$, i.e., $-3 < x < 1$. On the right, we show a plot of the function $f(x) = \frac{1}{1-x}$ (blue) and its power series representations $\sum_{k=0}^{\infty} x^k$ (red) and $\sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^k$ (green).



It is worth noting that in the previous example, both series diverge outside of the interval of convergence (even at the endpoints). Furthermore, for each x in the interval of convergence, the series converges to $f(x)$.

3 Taylor Series

Next, we consider the power series that arrise from the Taylor series of a function $f(x)$ at x_0.

3.1 Exponential and Logarithm functions

We have seen that the Taylor series of $f(x) = e^x$ at $x_0 = 0$ is given by

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Furthermore, this series converges absolute for all x and the value of the series is equal to e^x. Hence, we may write $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all real x.

Similarly, we have see that the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ is given by

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}.$$

Furthermore, this series converges absolutely for all x in $(0, 2)$, and converges conditionally for $x = 2$. For all x in $(0, 2]$, the value of the series is equal to $\ln(x)$; hence, we may write $\ln(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$ for all x in $(0, 2]$.

We will find it more useful to derive the Taylor series of $\ln(x + 1)$ and $\ln(x - 1)$ at $x_0 = 0$.

3.2 Triginometric Functions

Consider the function $f(x) = \sin(x)$ and $x_0 = 0$. Note that

$$f^{(k)}(0) = \begin{cases} 0, & 0 = k \mod 4 \\ 1, & 1 = k \mod 4 \\ 0, & 2 = k \mod 4 \\ -1, & 3 = k \mod 4 \end{cases}$$

Therefore, the Taylor series is given by

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Using the ratio test, we find that this series converges absolutely for all x. Furthermore, for any x, the remainder term for this Taylor series is bounded by

$$\left| \int_0^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \right| \leq \frac{|x|^{n+1}}{(n+1)!},$$

which converges to zero as $n \to \infty$. Therefore, we can write $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for all x.

4 Exercises

I. Derive the Taylor series of $\ln(x + 1)$ and $\ln(x - 1)$ at $x_0 = 0$. What is the interval of convergence for these series?

II. Derive the Taylor series of $\cos(x)$ at $x_0 = 0$.