Integration of Power Series

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1 Antiderivatives

Suppose that $f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$ has a positive radius of convergence $R > 0$. Define

$$F(x) = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-x_0)^{k+1},$$

for all $x \in (x_0 - R, x_0 + R)$. Note that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\left|c_{k+1}(x-x_0)^{k+2}\right|}{k+2} \cdot \frac{k+1}{\left|c_k(x-x_0)^{k+1}\right|} &= \lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|}|x-x_0| \\
&= \lim_{k\to\infty} \frac{\left|c_{k+1}(x-x_0)^{k+1}\right|}{\left|c_k(x-x_0)^{k}\right|},
\end{aligned}$$

so the limit of the ratios is the same for the series representing $f(x)$ and $F(x)$. Therefore, the function F is defined by the value of an absolutely convergent series. Furthermore, on the interval $(x_0 - R, x_0 + R)$, we can differentiate $F(x)$ as follows

$$\begin{aligned}
F'(x) &= \sum_{k=0}^{\infty} \frac{c_k}{k+1}(k+1)(x-x_0)^k \\
&= \sum_{k=0}^{\infty} c_k(x-x_0)^k = f(x).
\end{aligned}$$

Hence, $F(x)$ is an antiderivative of $f(x)$ on $(x_0 - R, x_0 + R)$.

2 Definite Integrals

Recall the limit definition of the definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where the interval $[a, b]$ is partitioned into n subintervals $[x_{i-1}, x_i]$, for $1 \leq i \leq n$, the length of the ith subinterval is Δx_i, and x_i^* is any point in the subinterval. Suppose that $f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$ has a positive radius of convergence $R > 0$. Also, suppose that the interval $[a, b]$ is contained in $(x_0 - R, x_0 + R)$. Then,

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sum_{k=0}^{\infty} c_k(x_i^* - x_0)^k \right) \Delta x_i \\
&= \sum_{k=0}^{\infty} \left(\lim_{n\to\infty} \sum_{i=1}^{n} c_k(x_i^* - x_0)^k \Delta x_i \right).
\end{aligned}$$

Note that the last equality only holds for absolutely convergent series. Furthermore,

$$\lim_{n\to\infty}\sum_{i=1}^{n} c_k(x_i^* - x_0)^k \Delta x_i = \int_a^b c_k(x-x_0)^k dx.$$

Therefore,

$$\int_a^b f(x)dx = \sum_{k=0}^{\infty}\int_a^b c_k(x-x_0)^k dx$$

We summarize the integral results for power series in the following theorem.

Theorem 2.1. *Let $f(x) = \sum_{k=0}^{\infty}$ have a positive radius of convergence $R > 0$. Then,*

$$F(x) = \sum_{k=0}^{\infty}\frac{c_k}{k+1}(x-x_0)^{k+1}$$

has radius of convergence R and is an antiderivative of $f(x)$ on $(x_0 - R, x_0 + R)$. Furthermore, if $[a, b]$ is a subset of $(x_0 - R, x_0 + R)$, then

$$\int_a^b f(x)dx = \sum_{k=0}^{\infty}\int_a^b c_k(x-x_0)^k dx.$$

As an example, consider the power series representation of the cosine function

$$f(x) = \cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}.$$

Then, we can represent an antiderivative of $f(x)$ as follows

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{x^{2k+1}}{2k+1} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\ &= \sin(x). \end{aligned}$$

Hence, $\sin(x)$ is an antiderivative of $\cos(x)$.

As another example, consider the following power series

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty}(-1)^{k+1}(x-2)^k,$$

which has a radius of convergence $R = 1$ and interval of convergence $(1, 3)$. Then, we can integrate $f(x)$ over $[3/2, 5/2]$ as follows

$$\begin{aligned} \int_{3/2}^{5/2}\frac{1}{1-x}dx &= \sum_{k=0}^{\infty}\int_{3/2}^{5/2}(-1)^{k+1}(x-2)^k dx \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(x-2)^{k+1}}{k+1}\Big|_{3/2}^{5/2} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+1}\left(\left(\frac{1}{2}\right)^{k+1} - \left(\frac{-1}{2}\right)^{k+1}\right) \\ &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\left(\frac{1}{2}\right)^{k} - \left(\frac{-1}{2}\right)^{k}\right) = \ln(1/2) - \ln(3/2). \end{aligned}$$