Calculus with Analytic Geometry II

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1 Power Series Operations

Last time, we used the power series

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k,\ 0 < x \leq 2$$

to determine $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$, for $-1 < x \leq 1$, and $\ln(1-x) = \sum_{k=1}^{\infty} -\frac{1}{k}x^k$, for $-1 \leq x < 1$. Furthermore, applying the subtraction operation gives us

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k + \sum_{k=1}^{\infty} \frac{1}{k} x^k \\
&= \sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1},\ -1 < x < 1.
\end{aligned}$$

Note that the interval of convergence is the intersection of the interval of convergence for $\ln(1+x)$ and $\ln(1-x)$.

We also used the power series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,\ -1 < x < 1$$

to determine $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, for $-1 < x < 1$, and $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$, for $-1 < x < 1$. Regarding the last interval of convergence, note that $x^2 < 1$ for all $-1 < x < 1$. Furthermore, applying the integral operation gives us

$$\begin{aligned}
\arctan(x) + C &= \int \frac{1}{1+x^2} dx \\
&= \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \\
&= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots.
\end{aligned}$$

Note that $C = 0$ since $\arctan(0) = 0$. Furthermore, the interval of convergence of a power series does not change under the derivative and integral test. Hence,

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1},$$

for all $-1 < x < 1$. In fact, since the given power series converges at $x = -1$ and $x = 1$, the continuity of $\arctan(x)$ implies that $\arctan(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ and $\arctan(-1) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$.

Finally, we consider the multiplication and division of power series. For instance, consider the rational function

$$\frac{1}{(x-1)(x-3)} = \frac{1}{x-1}\frac{1}{x-3}.$$

Note that

$$\frac{1}{x-1} = -\frac{1}{1-x} = -\sum_{k=0}^{\infty} x^k = -\left(1 + x + x^2 + x^3 + \cdots\right), \ -1 < x < 1$$

and

$$\frac{1}{x-3} = -\frac{1}{3}\frac{1}{1-x/3} = -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k = -\frac{1}{3}\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots\right), \ -3 < x < 3.$$

Therefore, the rational function can be represented as follows

$$\begin{aligned}\frac{1}{x-1}\frac{1}{x-3} &= \frac{1}{3}\left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots\right) \\ &= \frac{1}{3}\left(1 + x\left(\frac{1}{3} + 1\right) + x^2\left(\frac{1}{3^2} + \frac{1}{3} + 1\right) + x^3\left(\frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3} + 1\right) + \cdots\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right) x^k\end{aligned}$$

As an example of division, note that

$$\begin{aligned}\tan(x) &= \frac{\sin(x)}{\cos(x)} \\ &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}.\end{aligned}$$

Using long division, we find that

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$