Calculus with Analytic Geometry II

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1 The Ratio Test

Let $\sum_{k=1}^{\infty} u_k$ be a series, where u_k are positive for all $k \geq 1$. The ratio test defines

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}.$$

Then,

a. if $\rho < 1$, then the series converges.

b. if $\rho > 1$, then the series diverges.

c. if $\rho = 1$, then the ratio test is inclunclusive.

A proof of the ratio test (part a) follows.

Proof. Suppose that $\rho < 1$ and let $\rho < r < 1$. By definition of the limit, there is a positive integer N such that for all $k \geq N$, we have

$$\frac{u_{k+1}}{u_k} < r.$$

Therefore,

$$\begin{aligned}
u_{N+1} &< u_N r \\
u_{N+2} &< u_{N+1} r < u_N r^2 \\
&\vdots \\
u_{N+k} &< u_{N+k-1} r < \cdots < u_N r^k
\end{aligned}$$

□

The geometric series

$$\sum_{k=1}^{\infty} u_N r^k$$

converges since $0 < r < 1$. By the direct comparison test, the series $\sum_{k=1}^{\infty} u_{N+k}$ also converges. Furthermore, it follows that $\sum_{k=1}^{\infty} u_k$ must converge since discarding a finite number of terms does not affect convergence.

2 The Root Test

Let $\sum_{k=1}^{\infty} u_k$ be a series, where u_k are positive for all $k \geq 1$. The root test defines

$$\rho = \lim_{k\to\infty} u_k^{1/k}.$$

Then,

a. if $\rho < 1$, then the series converges.

b. if $\rho > 1$, then the series diverges.

c. if $\rho = 1$, then the ratio test is inclunclusive.

A proof of the root test (part a) follows.

Proof. Suppose that $\rho < 1$ and let $\rho < r < 1$. By definition of the limit, there is a positive integer N such that for all $k \geq N$, we have

$$u_k^{1/k} < r.$$

Therefore,

$$\begin{aligned} u_N &< r^N \\ u_{N+1} &< r^{N+1} \\ &\vdots \\ u_{N+k} &< r^{N+k} \end{aligned}$$

The geometric series

$$\sum_{k=N}^{\infty} r^k$$

converges since $0 < r < 1$. By the direct comparison test, the series $\sum_{k=N}^{\infty} u_k$ also converges. Furthermore, it follows that $\sum_{k=1}^{\infty} u_k$ must converge since discarding a finite number of terms does not affect convergence. □

3 Exercises

Use the root or ratio test to determine the convergence/divergence of the following series.

I. $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$

II. $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2k}}{k^k}$

III. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+1}{5k-2}\right)^k$

IV. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$