Calculus with Analytic Geometry II

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1 Sequences

Each time we have solved a geometric problem (area, volume, surface area), we have first approximated the solution as a sum. Then, we applied a limit to transform our sum into a Riemann sum, which we recognize as a definite integral. The mathematical concept underlying this methedology is known as a sequence. Today, we introduce the sequence in its general form. This investigation will lead us to the representation of functions via series, which is a powerful tool in the study of differential equations.

A sequence is a unending succession of numbers, called terms. It is understood that the terms have a definite order, that is, there is a first term a_1, a second term a_2, a third term a_3, and so on. Such a sequence can be written as

$$a_1,\ a_2,\ a_3,\ \ldots.$$

Due to this definite order, sequences can be viewed as functions from the positive integers to the real numbers. For example, the sequence

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots$$

can be written as the function $f(n) = \frac{n}{n+1}$. As another example, the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$$

can be written as the function $f(n) = \frac{1}{2^n}$. The function of a sequence can also be written using bracket notation: $\{f(n)\}_{n=1}^{\infty}$. For instance, the sequences above can be written as $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ and $\{\frac{1}{2^n}\}_{n=1}^{\infty}$, respectively.

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the value L if for any $\epsilon > 0$ there is a positive integer N such that $|a_n - L| < \epsilon$ for all $n > N$. In this case, we write $\lim_{n\to\infty} a_n = L$. For example, the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ converges to 1. Indeed, for each $\epsilon > 0$, Let $N = \frac{1-\epsilon}{\epsilon}$. Then, $n > N$ implies that $n > \frac{1-\epsilon}{\epsilon}$, i.e., $n + 1 > \frac{1}{\epsilon}$. Therefore,

$$\begin{aligned} \left|\frac{n}{n+1} - 1\right| &= \left|\frac{n-(n+1)}{n+1}\right| \\ &= \frac{1}{n+1} < \epsilon. \end{aligned}$$

If a sequence does not converge to a finite number L, then it is said to diverge. A sequence that diverges satisfies the following: For all finite numbers L, there exists an $\epsilon > 0$ such that for all positive integers N, there is a $n > N$ such that $|a_n - L| \geq \epsilon$. For example, the sequence $\{n\}_{n=1}^{\infty}$ diverges. Let L be any finite number and define $\epsilon = 1$. Then for any positive integer N, there is an $n > N$ such that $|n - L| \geq \epsilon$.

2 Series

An infinite series is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots.$$

To determine the convergence or divergence of an infinite series, we define the sequence of partial sums:

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} u_k \end{aligned}$$

If the sequence of partial sums $\{s_n\}$ converges to S, then the infinit series converges to S. If the sequence of partial sums diverges, then the infinite series diverges.

For example, consider the infinite series $\sum_{k=1}^{\infty} (-1)^k$, which has sequence of partial sums

$$\begin{aligned} s_1 &= -1 \\ s_2 &= 0 \\ s_3 &= -1 \\ s_4 &= 0 \\ &\vdots \end{aligned}$$

This sequence of partial sums diverges; hence, the infinite series diverges.

As another example, consider the infinite series $\sum_{k=0}^{n} \frac{1}{2^k}$, which has sequence of partial sums

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_2 &= \frac{3}{2} + \frac{1}{4} = \frac{7}{4} \\ s_3 &= \frac{7}{4} + \frac{1}{8} = \frac{15}{8} \\ &\vdots \\ s_n &= \frac{2^{n+1} - 1}{2^n}, \end{aligned}$$

which converges to 2.

As another example, consider the series $\sum_{k=1}^{n} \frac{1}{k(k+1)}$, which has sequence of partial sums

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ s_3 &= \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\ s_4 &= \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \\ &\vdots \\ s_n &= \frac{n}{n+1}, \end{aligned}$$

which converges to 1.