Calculus with Analytic Geometry II

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February 17, 2025

1 Infinite Series

An infinite series is an expression that can be written in the form

$$\sum_{i=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots .$$

To determine the convergence or divergence of an infinite series, we define the sequence of partial sums:

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} u_k \end{aligned}$$

If the sequence of partial sums $\{s_n\}$ converges to S, then the infinit series converges to S. If the sequence of partial sums diverges, then the infinite series diverges.

For example, consider the infinite series $\sum_{k=1}^{\infty} (-1)^k$, which has sequence of partial sums

$$\begin{aligned} s_1 &= -1 \\ s_2 &= 0 \\ s_3 &= -1 \\ s_4 &= 0 \\ &\vdots \end{aligned}$$

This sequence of partial sums diverges; hence, the infinite series diverges.

As another example, consider the infinite series $\sum_{k=0}^{n} \frac{1}{2^k}$, which has sequence of partial sums

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_2 &= \frac{3}{2} + \frac{1}{4} = \frac{7}{4} \\ s_3 &= \frac{7}{4} + \frac{1}{8} = \frac{15}{8} \\ &\vdots \\ s_n &= \frac{2^{n+1} - 1}{2^n} \end{aligned}$$

Note that

$$\lim_{n\to\infty} \frac{2^{n+1}-1}{2^n} = 2,$$

so the given infinite series converges to 1. The previous example, is an instance of a geometric series. In general, a geometric series can be written as

$$\sum_{k=0} ar^k,$$

where a and r are non-zero. If $|r| \geq 1$, then the series diverges. If $|r| < 1$, then the series converges to $\frac{a}{1-r}$.

As another example, consider the series $\sum_{k=1}^{n} \frac{1}{k(k+1)}$, which has sequence of partial sums

$$\begin{aligned}
s_1 &= \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\
s_3 &= \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\
s_4 &= \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \\
&\vdots \\
s_n &= \frac{n}{n+1}
\end{aligned}$$

Note that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1,$$

so the given infinite series converges to 1. The previous example is an instance of a telescoping series. Indeed, using partial fractional decomposition we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\
&= 1.
\end{aligned}$$

2 Exercises

For each series, determine if the series converges or diverges. If the series converges, find its limiting value.

I. $\displaystyle\sum_{k=0}^{\infty} \frac{5}{4^k}$

II. $\displaystyle\sum_{k=0}^{\infty} 3^{2k} 5^{1-k}$

III. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1}$