Calculus with Analytic Geometry II

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1 Geometric Series

A geometric series is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots,$$

where a and r are non-zero. Consider the nth partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Then, multiplication by r yields

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Therefore,

$$rs_n - s_n = ar^n - a.$$

So, we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{r-1}\left(r^n - 1\right),$$

which converges to $\frac{a}{1-r}$ if and only if $|r| < 1$.

As an example, consider the series $\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k$. This is clearly a geometric series with $a = 1$ and $r = 1/10$. Hence, this series converges to

$$\frac{1}{9/10} = \frac{10}{9} = 1 + \frac{1}{9} = 1.1111\ldots.$$

2 Telescoping Series

A telescoping series can be written in the form

$$\sum_{k=1}^{\infty}\left(b_k - b_{k+1}\right) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots.$$

Note that b_2, b_3, b_4, and so on will cancel. Hence, this series converges to b_1.

As an example, consider the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right),$$

which clearly converges to 1. As another example, consider the series

$$\sum_{k=2}^{\infty}\frac{1}{k^2 - 1}.$$

A partial fraction decomposition to yields

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{1}{k^2-1} &= \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots \right) \\
&= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}.
\end{aligned}$$

3 Tests for Divergence

In certain cases, it is easy to identify when a series diverges. In particular, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ must diverge. For example, the series $\sum_{k=1}^{\infty} k$ diverges.

However, $\lim_{k\to\infty} = 0$ is not enough to gurantee convergence. The most famous example is the harmonic series, $\sum_{k=1}^{\infty} \frac{1}{k}$. Consider the following partial sums:

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
s_4 &= \frac{3}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \\
s_8 &= \frac{25}{12} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{3943}{840} \\
&\vdots \\
s_{2^n} &= s_{2^{n-1}} + \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n} \\
&> s_{2^{n-1}} + \frac{2^{n-1}}{2^n} = s_{2^{n-1}} + \frac{1}{2}.
\end{aligned}$$

Therefore, $s_{2^n} - s_{2^{n-1}} > 1/2$ for all $n \geq 1$, which implies that the harmonic series diverges.