Calculus with Analytic Geometry II

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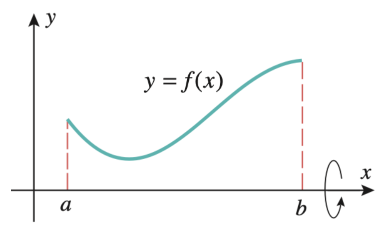
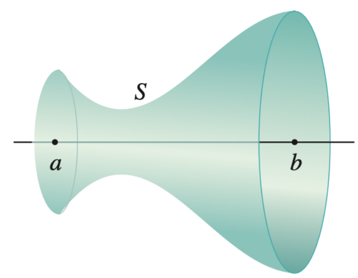
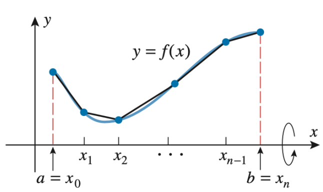
1 Surface of Revolution

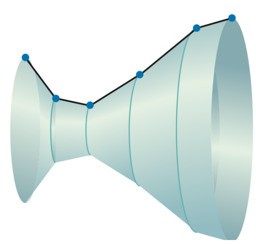


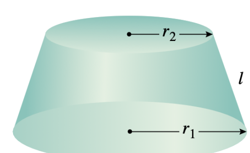
Let $f(x)$ denote a smooth function on $[a, b]$ and let R denote the plane region bounded by $y = f(x)$ and the x-axis over $[a, b]$. Consider the solid of revolution formed by revolving R about the x-axis, and let S denote the surface area of this solid.



To approximate S we split the interval $[a, b]$ into n subintervals each of length Δx. Over each subinterval $[x_{k-1}, x_k]$, we approximate the curve $y = f(x)$ via the line segment connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. When these line segments are revolved about the x-axis, it generates a surface consisting of n parts, each of which is a portion of a right circular cone.







The area of each approximating surface is given by

$$S_k = 2\pi \left(\frac{f(x_{k-1}) + f(x_k)}{2} \right) L_k,$$

where L_k denotes the length of the line segment connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. Recall that

$$L_k = \Delta x \sqrt{1 + f'(x_k^*)^2},$$

where x_k^* is in $[x_{k-1}, x_k]$ such that $f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x$, guaranteed by the mean value theorem. Furthermore, since $f(x)$ is continuous, the intermediate value theorem implies that there exists a x_k^{**} in

$[x_{k-1}, x_k]$ such that

$$\frac{f(x_{k-1}) + f(x_k)}{2} = f(x_k^{**}).$$

Therefore, the surface area S can be approximated via

$$S \approx 2\pi \sum_{i=1}^{n} f(x_i^{**}) \sqrt{1 + f'(x_i^*)^2} \Delta x.$$

Taking the limit as $n \to \infty$ gives us

$$S = \lim_{n \to \infty} 2\pi \sum_{i=1}^{n} f(x_i^{**}) \sqrt{1 + f'(x_i^*)^2} \Delta x = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx.$$

2 Exercises

I. Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

II. Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.