Calculus with Analytic Geometry II

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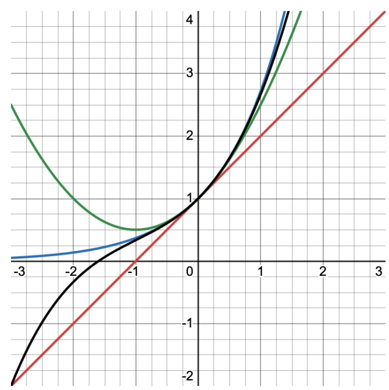
1 Taylor Polynomials

Suppose that $f(x)$ is n-times differentiable at x_0. Then the nth Taylor polynomial of $f(x)$ at x_0 is

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \cdots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!}$$

For example, let $f(x) = e^x$ and $x_0 = 0$. Then, the $n = 1, 2, 3$ Taylor polynomials of $f(x)$ at x_0 are shown below:

$$\begin{aligned} p_1(x) &= 1 + x \\ p_2(x) &= 1 + x + \frac{x^2}{2} \\ p_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned}$$



The plot of $f(x)$ (blue), $p_1(x)$ (red), $p_2(x)$ (green), and $p_3(x)$ (black) are shown on the right. Note that all Taylor polynomials agree with $f(x)$ at x_0. Further, the first derivative of all Taylor polynomials agree with $f'(x)$ at x_0. The second derivative of $p_2(x)$ agrees with $f''(x)$ at x_0 and the third derivative of $p_3(x)$ agrees with $f'''(x)$ at x_0.

In general we have the following result regarding the value of $p_n(x)$ and its derivatives at x_0.

Theorem 1.1. *Suppose that $f(x)$ is n-times differentiable at x_0 and let $p_n(x)$ denote the nth Taylor polynomial of $f(x)$ at x_0. Then,*

$$f^{(k)}(x_0) = p_n^{(k)}(x_0),$$

for all $0 \leq k \leq n$.

2 Taylor Polynomial Remainder

We can use the Taylor polynomial to approximate a function. Moreover, we can bound the error in the Taylor polynomial approximation. To this end, note that

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt.$$

Applying integration by parts,

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(t)dt \\ &= f(x_0) + (xf'(x) - x_0 f'(x_0)) - \int_{x_0}^{x} tf''(t)dt \\ &= f(x_0) + x\left(f'(x_0) + \int_{x_0}^{x} f''(t)dt\right) - x_0 f'(x_0) - \int_{x_0}^{x} tf''(t)dt \\ &= f(x_0) + (x - x_0)f'(x_0) + \int_{x_0}^{x} (x-t)f''(t)dt \end{aligned}$$

Next, we generalize the integral remainder formula for any $n \geq 1$ using induction. Let $n \geq 1$ and suppose that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

Applying integration by parts,

$$\int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt = \frac{(x-x_0)^n}{n!} + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Therefore, for any $n \geq 1$, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + f^{(n)}(x)\frac{(x-x_0)^n}{n!} + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Suppose that $f\left|^{(n+1)}(t)\right| \leq M$, for all t in the interval $[x_0, x]$, then

$$|f(x) - p_n(x)| = \left|\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt\right| \leq M\frac{|x-x_0|^{n+1}}{(n+1)!}$$

As an example, we will use the Taylor series of $f(x) = e^x$ at $x_0 = 0$ to approximate e to 2-decimal places. To this end, note that all derivatives of $f(x)$ are bounded above by e on the interval $[0, 1]$. Hence, the error bound int he Taylor series approximation is given by

$$e\frac{|x|^{n+1}}{(n+1)!} \leq \frac{e}{(n+1)!},$$

for all x in the interval $[0, 1]$. To guarantee 2-decimal places of accuracy, we need $\frac{e}{(n+1)!} \leq 0.005$, i.e.,

$$(n+1)! \geq \frac{e}{0.005} = 500e.$$

Note that $7! = 5040$, which is significantly bigger than $500e$. Hence, $n = 6$ is sufficient for our Taylor series approximation. In conclusion, the approximation of e given by the $n = 6$th Taylor series approximation of $f(x) = e^x$ at $x_0 = 0$ is

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.7181.$$

Which is exact up to the 4th decimal place.