Taylor Series

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1 Taylor Series

Suppose that $f(x)$ is infinitely differentiable on a interval containing x_0 and let $p_n(x)$ denote the nth Taylor polynomial of $f(x)$ at x_0. Then, for all x in that interval, we have

$$f(x) = p_n(x) + R_n(x),$$

where $R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$ is the remainder term. If $\lim_{n\to\infty} R_n(x) = 0$, then

$$f(x) = \lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

where the series is known as the Taylor series of $f(x)$.

As an example, consider the function $f(x) = e^x$ and center $x_0 = 0$. The Taylor polynomial is given by $p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ with remainder term

$$R_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$

Since $f^{(n+1)}(t) = e^t$ is an increasing function, the remainder term can be bounded as follows:

$$|R_n(x)| \leq \begin{cases} e^x \frac{|x|^{n+1}}{(n+1)!} & x \geq 0, \\ \frac{|x|^{n+1}}{(n+1)!} & x < 0. \end{cases}$$

Hence, $\lim_{n\to\infty} R_n(x) = 0$ for all x. Therefore, we have the following Taylor series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad -\infty < x < +\infty.$$

As another example, consider the function $f(x) = \sin(x)$ and center $x_0 = 0$. The Taylor polynomial is given by $p_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ with remainder term

$$R_{2n+1}(x) = \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} f^{(2n+2)}(t)dt.$$

Note that $\left|f^{(2n+2)}(t)\right| = |\sin(t)| \leq 1$ for all $n \geq 0$. Therefore, the remainder term can be bounded as follows:

$$|R_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

Hence, $\lim_{n\to\infty} R_n(x) = 0$ for all x. Therefore, we have the following Taylor series for $\sin(x)$:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \qquad -\infty < x < +\infty.$$

As a final example, consider the function $f(x) = \ln(x)$ and center $x_0 = 0$. The Taylor polynomial is given by $p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$ with remainder term

$$R_n(x) = \int_1^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

Since $f^{(n+1)}(t) = (-1)^n \frac{n!}{t^{n+1}}$, the remainder term becomes

$$R_n(x) = (-1)^n \int_1^x \frac{(x-t)^n}{t^{n+1}} dt.$$

For $1 \le x \le 2$, we have $t \ge 1$, so

$$|R_n(x)| \le \frac{|x-t|^{n+1}}{n+1}.$$

Since $1 \le x \le 2$ and $1 \le t \le x$, it follows that $|x-t| \le 1$. Therefore, $\lim_{n\to\infty} R_n(x) = 0$.

For $0 < x < 1$, we use the substitution $u = \frac{t-x}{t}$, i.e., $t = \frac{x}{1-u}$, to rewrite the remainder as

$$R_n(x) = \int_1^x \frac{(t-x)^n}{t^{n+1}} dt = \int_{1-x}^0 \frac{u^n}{1-u} du.$$

Therefore, the remainder is bounded above by

$$|R_n(x)| = \int_0^{1-x} \frac{u^n}{1-u} du \le \frac{1}{x} \int_0^{1-x} u^n du = \frac{1}{x} \frac{(1-x)^{n+1}}{n+1}.$$

Since $0 < x < 1$, it follows that $\lim_{n\to\infty} R_n(x) = 0$.

It follows that $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in (0, 2]$. Therefore, we have the following Taylor series for $\ln(x)$:

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k, \qquad 0 < x \le 2.$$